

# THE HOMOTOPY CATEGORY FOR GENTLE ALGEBRAS.

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**ABSTRACT.** We consider possibly infinite dimensional gentle algebras defined by taking the completion a locally-gentle algebra with respect to the arrow ideal. We show the unbounded homotopy category of finitely generated projective modules satisfies the Krull-Remak-Schmidt property, and that complexes here are direct sums of *string* and *band* complexes. The aim of this work is to investigate new situations one may adapt a classification technique, sometimes called the functorial filtration method. The name refers to the construction of certain functorially defined filtrations which are used in proving correctness.

## 1. INTRODUCTION.

In this article we classify the objects of the unbounded homotopy category  $\mathcal{K}(\overline{\Lambda} - \mathbf{proj})$  of finitely generated projective left modules, where  $\overline{\Lambda}$  is a gentle algebra defined as follows.

$k$  will be a field and  $Q$  a finite quiver with path algebra  $kQ$ . The *completed path algebra*  $\overline{kQ}$  of  $Q$  consists of possibly infinite sums  $\sum \lambda_\sigma \sigma$  where  $\sigma$  runs through the set of all paths and the elements  $\lambda_\sigma$  are scalars from  $k$ . In general it is possible to have  $\lambda_\sigma \neq 0$  for infinitely many  $\sigma$ , say when  $Q$  has linearly orientated cycles.  $\overline{kQ}$  is a  $k$ -algebra where addition and multiplication are defined so that  $kQ = \overline{kQ}$  if  $Q$  has no linearly oriented cycles (see [16] for details). For example the power series ring  $k[[x]]$  is the completed path algebra of the quiver consisting of a loop.

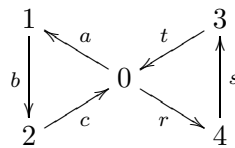
**Gentle algebras.** By a *gentle algebra* we mean an algebra of the form  $\overline{\Lambda} = \overline{kQ}/(\rho)$  where  $(\rho)$  is the ideal of  $\overline{kQ}$  generated by a set  $\rho$ , and

- (I) any vertex is the head of at most two arrows and the tail of at most two arrows,
- (II) any element of  $\rho$  is a path of length 2,  
and given  $\beta$  is an arrow,
- (III) there is at most one path  $\alpha\beta$  with  $\alpha\beta \in \rho$  and at most one path  $\beta\gamma$  with  $\beta\gamma \in \rho$ ,
- (IV) there is at most one path  $\alpha'\beta$  of length 2 with  $\alpha'\beta \notin \rho$  and at most one path  $\beta\gamma'$  of length 2 with  $\beta\gamma' \notin \rho$ .

**Example (GP).** The quotient  $k[[x, y]]/(xy)$  of the power series ring in two variables is gentle, where the quiver consists of two loops labeled  $x$  and  $y$  at one vertex, and  $\rho = \{xy, yx\}$ . The reader is encouraged to keep this example in mind. The finite dimensional representations of this algebra were classified by Gel'fand and Ponomarev [22]. An exposition on this work was given by Gabriel [20] who went on to generalise this idea [19].

**Remark.** Let  $(\rho)$  be the ideal in  $kQ$  generated by  $\rho$ . Bessenrodt and Holm [7] call the algebra  $\Lambda = kQ/(\rho)$  a locally-gentle algebra. If there exists some  $n > 0$  for which  $A^n \subseteq (\rho)$  where  $A$  is the ideal of  $kQ$  generated by the arrows, then  $\Lambda = \overline{\Lambda}$ . Consequently the definition above subsumes the finite dimensional gentle algebras introduced by Assem and Skowroński [2].

**Example (ALP).** Arnesen, Laking and Paukstello [1] consider a *running example*, the finite dimensional gentle algebra  $\Gamma = kQ/(\rho)$  where  $\rho = \{ba, cb, ac, sr, ts, rt\}$  and  $Q$  is the quiver



This example is used repeatedly in [1] to calculate a basis for the morphism space between indecomposable complexes. It is a nice example to observe non-trivial phenomena.

**Notation.** One can equivalently define  $\overline{\Lambda}$  as the completion of  $\Lambda$  with respect to the  $A$ -adic topology. This is used in section 8 which is why we use the over-line notation.  $\overline{\Lambda} - \mathbf{Mod}$  will be the category of all modules and  $\overline{\Lambda} - \mathbf{mod}$ ,  $\overline{\Lambda} - \mathbf{Proj}$  and  $\overline{\Lambda} - \mathbf{proj}$  will be the full subcategories of finitely generated, projective, and finitely generated projective modules respectively. If  $\mathcal{A}$  is any of the above categories then  $\mathcal{C}(\mathcal{A})$  will be the category of complexes (of objects in  $\mathcal{A}$ ),  $\mathcal{K}(\mathcal{A})$  the homotopy category and  $\mathcal{D}(\mathcal{A})$  the derived category. The notation  $M^\bullet$  will be used for a complex. The usual notation for full subcategories of bounded above/below complexes will be used. For example  $\mathcal{K}^{-b}(\overline{\Lambda} - \mathbf{proj})$  is the full subcategory (of the homotopy category  $\mathcal{K}(\overline{\Lambda} - \mathbf{Mod})$ ) consisting of complexes of finitely generated projectives which are right bounded with bounded homology.

**Definition 1.1.** For a complex  $M^\bullet$  (in  $\mathcal{A}$ ) we say  $M^\bullet$  has radical images if  $\mathrm{im}(d_M^r) \subseteq \mathrm{rad}(M^{r+1})$  for each  $r \in \mathbb{Z}$ . Let  $\mathcal{C}_{\mathrm{rad}}(\mathcal{A})$  denote the full subcategory of  $\mathcal{C}(\mathcal{A})$  consisting of complexes with radical images. Similarly let  $\mathcal{K}_{\mathrm{rad}}(\mathcal{A})$  denote the full subcategory of  $\mathcal{K}(\mathcal{A})$  consisting of complexes with radical images.

In order to study complexes of projectives one can start by using that a bounded complex  $M^\bullet$  of finitely generated projectives (over an appropriate ring) is homotopy equivalent to a complex  $N^\bullet$  with radical images. For example this was used in [4] and [5] where  $N^\bullet$  is called a minimal resolution. The construction of  $N^\bullet$  is standard, and involves taking projective covers. This may be done in our setting, as in section 7 we show that  $\overline{\Lambda}$  is complete with respect to its radical, and hence  $\overline{\Lambda}$  is *semi-perfect* (see [26, p. 336, Definition 23.1] and [26, p. 362, Proposition 24.12]). More precisely, at the end of the appendix we prove the following.

**Proposition 1.1.** *Any object of  $\mathcal{K}(\overline{\Lambda} - \mathbf{proj})$  is isomorphic to an object in  $\mathcal{K}_{\mathrm{rad}}(\overline{\Lambda} - \mathbf{proj})$ .*

Bekkert and Merklen [5] have classified the indecomposables of  $\mathcal{D}^b(\Lambda - \mathbf{mod})$  in the case  $\Lambda$  is a finite dimensional gentle algebra. The method of classification they use is sometimes called matrix reductions, which involves constructing a functor (preserving and respecting indecomposables) to the category of representations of a poset, which Bondarenko and Drozd [8] have shown reduces to a matrix problem solved by Nazarova and Roiter [28, 29, 30, 31, 32].

Bekkert, Drozd and Furtorny [6, p.2440, Theorem 2.7] note that with minor modifications, the matrix reduction used in [5] works in the case of the gentle algebras we are considering. Hence a classification of objects in  $\mathcal{D}^b(\overline{\Lambda} - \mathbf{mod})$  is not new. Burban and Drozd [11] proposed a matrix reduction for  $\mathcal{D}^-(\Lambda - \mathbf{mod})$  in the case  $\Lambda$  is a finite dimensional gentle algebra. In section 8 we classify the indecomposables in  $\mathcal{D}^-(\overline{\Lambda} - \mathbf{mod})$ , recovering these results.

Here we use a different approach to solving our classification problem, which we refer to in the abstract as the functorial filtration method. This method has been used several times [13, 14, 15, 18, 22, 33, 36] to prove correctness in certain module classifications. The purpose of this article is to highlight the functorial filtration method as a way to classify complexes of projective modules up to homotopy.

The functorial filtration method yields additional insight; parts (a) and (b) of theorem 6.1 identify the summands of a direct sum of string and band complexes, parts (c) and (d) of theorem 6.1 counts how many indecomposable summands have a non-zero homogeneous component in a chosen degree, theorem 1.1 shows all string complexes are indecomposable, and together with theorems 1.2 and 1.3 we obtain a classification of the homotopy category of unbounded complexes of finitely generated projectives, which together with lemma 8.2 recovers the calculation of the singularity category by Kalck [24, p.3, Theorem 2.5 (b)].

**Words.** Words have been used before to classify objects in terms of *strings* and *bands*. Crawley-Boevey [13] used infinite (and finite) words in an alphabet of arrows to show finitely controlled and point-wise Artinian modules over a string algebra are direct sums of string modules and (finite dimensional or primitive injective) band modules. *Generalised strings and bands* were used in [5]. These were defined by an alphabet of paths, and these words are called *homotopy strings and bands* in [1].

In what follows we create a new word system which adapts the functorial filtration method to our setting. To do so we use a modified version of the alphabet used in [5], and use the conventions on infinite words and their composition used in [13].

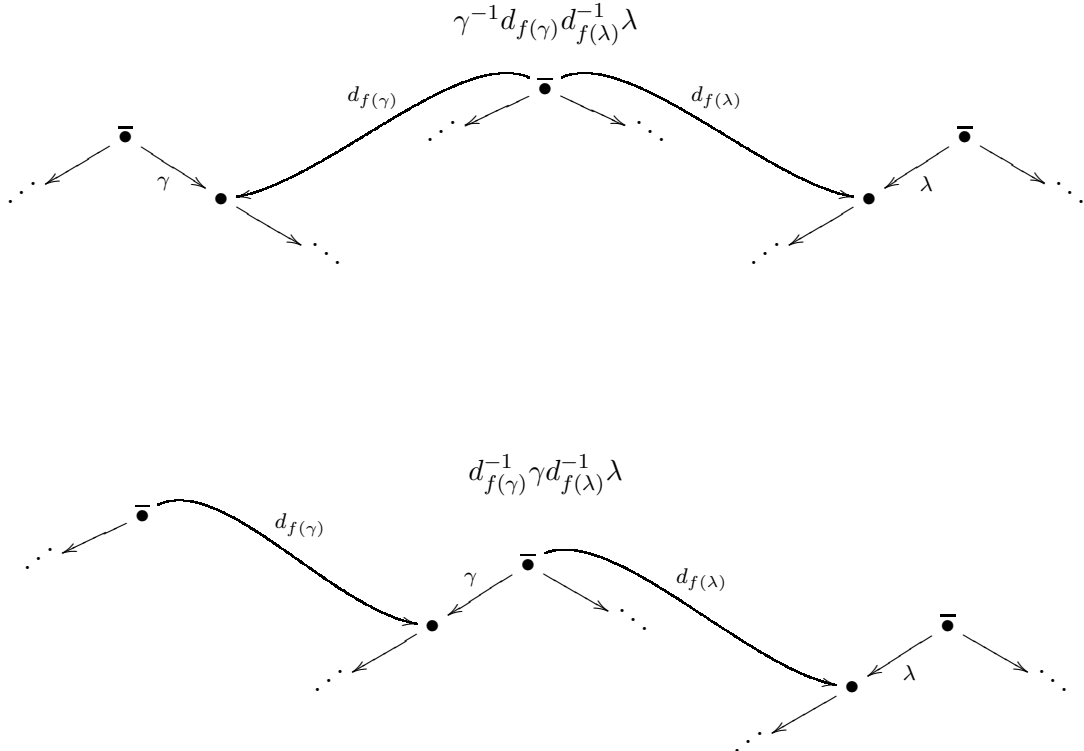
Any non-trivial path  $\gamma$  has a first arrow  $f(\gamma)$  and a last arrow  $l(\gamma)$  satisfying  $f(\gamma)\gamma' = \gamma''l(\gamma)$  for some paths  $\gamma'$  and  $\gamma''$ . Let  $\mathbf{Pa}_\rho$  the set of non-trivial paths which do not factor through an element of  $(\rho)$  (that is, the set of non-trivial paths which define non-zero cosets in  $\overline{\Lambda}$ ). A *letter*  $q$  is one of  $\gamma$ ,  $\gamma^{-1}$ ,  $d_\alpha$ , or  $d_\alpha^{-1}$  for  $\gamma \in \mathbf{Pa}_\rho$  and an arrow  $\alpha$ . The *inverse*  $q^{-1}$  of a letter  $q$  is defined by setting  $(\gamma)^{-1} = \gamma^{-1}$ ,  $(\gamma^{-1})^{-1} = \gamma$ ,  $(d_\alpha)^{-1} = d_\alpha^{-1}$  and  $(d_\alpha^{-1})^{-1} = d_\alpha$ . Let  $I$  be one of the sets  $\{0, \dots, m\}$  (for some  $m \geq 0$ ),  $\mathbb{N}$ ,  $-\mathbb{N} = \{-n \mid n \in \mathbb{N}\}$ , or  $\mathbb{Z}$ . For  $I \neq \{0\}$  an  $I$ -word is a sequence of letters

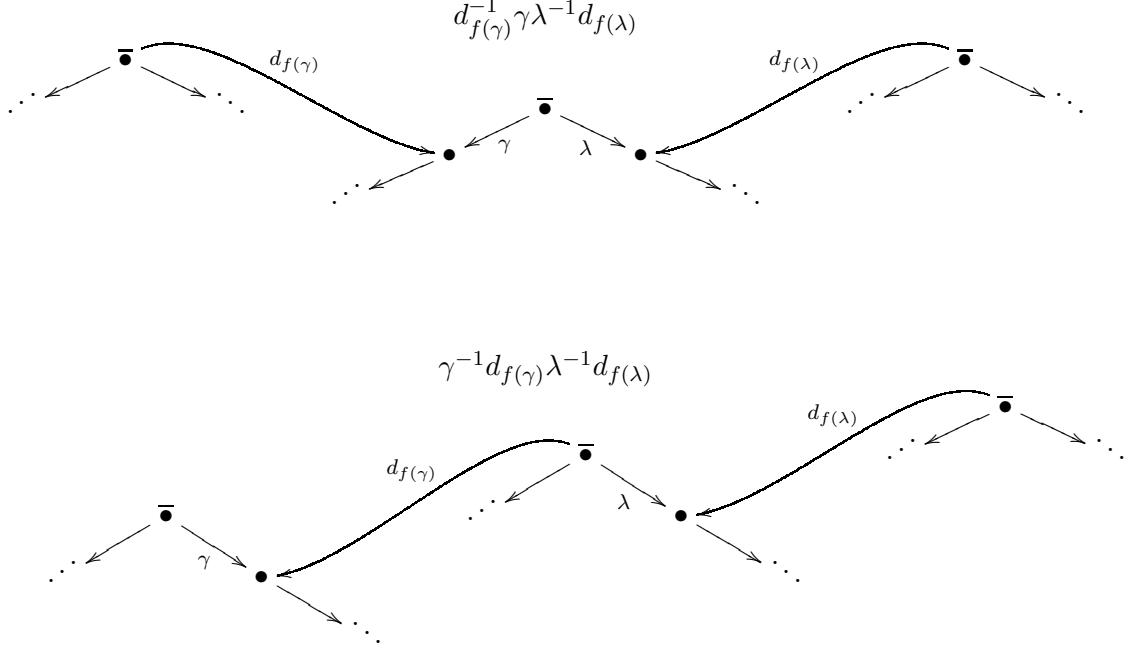
$$C = \begin{cases} l_1^{-1}r_1 \dots l_m^{-1}r_m & (\text{if } I = \{0, \dots, m\}) \\ l_1^{-1}r_1 l_2^{-1}r_2 \dots & (\text{if } I = \mathbb{N}) \\ \dots l_{-1}^{-1}r_{-1} l_0^{-1}r_0 & (\text{if } I = -\mathbb{N}) \\ \dots l_{-1}^{-1}r_{-1} l_0^{-1}r_0 \mid l_1^{-1}r_1 l_2^{-1}r_2 \dots & (\text{if } I = \mathbb{Z}) \end{cases}$$

(written as  $C = \dots l_i^{-1}r_i \dots$  to save space) such that any pair of letters  $l_i^{-1}r_i$  is one of  $\gamma^{-1}d_{f(\gamma)}$  or  $d_{f(\gamma)}^{-1}\gamma$  for some  $\gamma \in \mathbf{Pa}_\rho$ , and any consecutive pairs of letters  $l_i^{-1}r_i l_{i+1}^{-1}r_{i+1}$  is one of

$$\begin{aligned} & \gamma^{-1}d_{f(\gamma)}d_{f(\lambda)}^{-1}\lambda && (\text{where } h(\gamma) = h(\lambda) \text{ and } f(\gamma) \neq f(\lambda)) \\ & d_{f(\gamma)}^{-1}\gamma d_{f(\lambda)}^{-1}\lambda && (\text{where } l(\gamma)f(\lambda) \in \rho) \\ & d_{f(\gamma)}^{-1}\gamma \lambda^{-1}d_{f(\lambda)} && (\text{where } t(\gamma) = t(\lambda) \text{ and } l(\gamma) \neq l(\lambda)) \\ & \text{or } \gamma^{-1}d_{f(\gamma)}\lambda^{-1}d_{f(\lambda)} && (\text{where } l(\lambda)f(\gamma) \in \rho) \end{aligned}$$

For  $I = \{0\}$  *trivial words* will refer to the symbols  $1_{v,1}$  and  $1_{v,-1}$  for each vertex  $v$ . For each possibility of  $l_i^{-1}r_i l_{i+1}^{-1}r_{i+1}$  the reader is encouraged to draw the respective schema as indicated.





Elements of  $I$  correspond to the nodes  $\bar{\bullet}$  which will come to symbolize the heads of indecomposable projective  $\bar{\Lambda}$ -modules. The nodes  $\bar{\bullet}$  and  $\bullet$  together will index the *heads* and *tails* of letters, which are yet to be defined.

The pairs  $l^{-1}r$  that our words decompose into are in bijective correspondence with the alphabet used in [5]. In the next section we give meaning and motivation for the use of the letters  $d_\alpha$ . We now introduce some book-keeping, defined for a fixed yet arbitrary  $I$ -word  $C$ .

**Inverting words.** As usual  $C$  has an *inverse* defined by  $(1_{v,\delta})^{-1} = 1_{v,-\delta}$  for trivial words, and otherwise inverting the letters and reversing their order. So the inverse of an  $\mathbb{N}$  word is a  $-\mathbb{N}$  word and if  $I$  is (finite or  $\mathbb{Z}$ ) then the inverse of an  $I$  word is again an  $I$  word. Note the  $\mathbb{Z}$ -words are indexed so that

$$(\dots l_{-1}^{-1} r_{-1} l_0^{-1} r_0 \mid l_1^{-1} r_1 l_2^{-1} r_2 \dots)^{-1} = \dots r_2^{-1} l_2 r_1^{-1} l_1 \mid r_0^{-1} l_0 r_{-1}^{-1} l_{-1} \dots$$

**Vertices between letters.** The head and tail of any path  $\gamma \in \mathbf{Pa}_\rho$  are already defined and we extend this notion to all letters by setting  $h(d_a^{\pm 1}) = h(a)$  and  $h(q^{-1}) = t(q)$ .

For each  $i \in I$  there is an associated vertex  $v_C(i)$  defined by  $v_C(i) = t(l_{i+1})$  for  $i \leq 0$  and  $v_C(i) = t(r_i)$  for  $i > 0$  provided  $C = \dots l_i^{-1} r_i \dots$  is non-trivial, and  $v_{1_{v,\pm 1}}(0) = v$  otherwise.

**Homogeny.** Let  $H(\gamma^{-1} d_{f(\gamma)}) = -1$  and  $H(d_{f(\gamma)} \gamma) = 1$  for any  $\gamma \in \mathbf{Pa}_\rho$ .

Define  $\mu_C : I \rightarrow \mathbb{Z}$  defined by sending 0 to 0,  $i > 0$  to  $\sum_{t=1}^i H(l_t^{-1} r_t)$  and  $j < 0$  to  $-\sum_{t=j+1}^0 H(l_t^{-1} r_t)$ .

We say  $C$  has *controlled homogeny* if  $\mu_C^{-1}(t) = \{i \in I \mid \mu_C(i) = t\}$  is a finite set for each  $t \in \mathbb{Z}$ .

**Complexes of the form  $P^\bullet(C)$ .** Let  $P(C) = \bigoplus_{n \in \mathbb{Z}} P^n(C)$  where for each  $n \in \mathbb{Z}$  we let  $P^n(C)$  be the direct sum  $\bigoplus_i \bar{\Lambda} e_{v_C(i)}$  where  $i$  runs through the pre-image  $\mu_C^{-1}(n)$ . Hence  $C$  has controlled homogeny if and only if  $P^n(C)$  is finitely generated for each  $n \in \mathbb{Z}$ .

For each  $i \in I$  let  $b_i$  denote the coset of  $e_{v_C(i)}$  in the summand  $\bar{\Lambda} e_{v_C(i)}$  of  $P^{\mu_C(i)}(C)$ .

We define the complex  $P^\bullet(C)$  by extending the assignment  $d_{P(C)}(b_i) = b_i^- + b_i^+$  linearly for each  $i \in I$  where

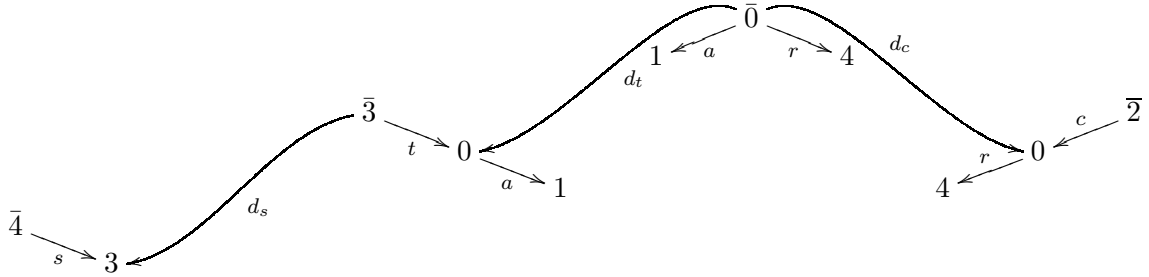
$$b_i^+ = \begin{cases} \alpha b_{i+1} & (\text{if } i+1 \in I \text{ and } l_{i+1}^{-1} r_{i+1} = d_{f(\alpha)} \alpha) \\ 0 & (\text{otherwise}) \end{cases}$$

$$b_i^- = \begin{cases} \beta b_{i-1} & (\text{if } i-1 \in I \text{ and } l_i^{-1} r_i = \beta^{-1} d_{f(\beta)}) \\ 0 & (\text{otherwise}) \end{cases}$$

**Example (ALP).** Recall the running example from [1]. Indecomposable projective modules here have the form to  $\Gamma e_i$  for some  $i = 0, 1, 2, 3, 4$ . The  $\{0, 1, 2, 3\}$  word  $D = s^{-1}d_s t^{-1}d_t d_c^{-1}c$  defines the complex  $P^\bullet(D)$  depicted by

$$\begin{array}{ccccccc}
 & & \Gamma e_0 & & P^{-2}(D) & & \\
 & \swarrow t & & \searrow c & & \downarrow d_{P(D)}^{-2} & \\
 & & \Gamma e_3 & & \Gamma e_2 & & P^{-1}(D) \\
 & \swarrow s & & & & \downarrow d_{P(D)}^{-1} & \\
 \Gamma e_4 & & & & & & P^0(D)
 \end{array}$$

We now draw the schemas that depict each projective indecomposable. This is to elucidate the correspondence between the word and the complex.



**Example (GP).** Recall the gentle algebra  $\bar{\Lambda} = k[[x, y]]/(xy)$ . Write  $x^{-m}$  and  $y^{-m}$  for  $(x^m)^{-1}$  and  $(y^m)^{-1}$  for each  $n, m > 0$ . Then the word

$$C = x^{-2}d_x y^{-1}d_y x^{-1}d_x d_y^{-1}y^3d_x^{-1}xy^{-1}d_y x^{-2}d_x y^{-1}d_y x^{-2}d_x \dots$$

is an example of an  $\mathbb{N}$ -word.  $P^\bullet(C)$  may be depicted by

$$\begin{array}{ccccccc}
 & & & & \bar{\Lambda} & \xleftarrow{x^2} & \dots \\
 & & & & \downarrow y & & \vdots \\
 & & \bar{\Lambda} & \xleftarrow{y^3} & \bar{\Lambda} & \xleftarrow{x^2} & P^{-4}(C) \\
 & & \downarrow x & & \downarrow y & & \downarrow d_{P(C)}^{-4} \\
 & & \bar{\Lambda} & \xleftarrow{x} & \bar{\Lambda} & \xleftarrow{x^2} & P^{-3}(C) \\
 & & \downarrow y & & \downarrow x & & \downarrow d_{P(C)}^{-3} \\
 & & \bar{\Lambda} & \xleftarrow{y} & \bar{\Lambda} & \xleftarrow{y} & P^{-2}(C) \\
 & & \downarrow x^2 & & \downarrow y & & \downarrow d_{P(C)}^{-2} \\
 & & \bar{\Lambda} & \xleftarrow{x^2} & \bar{\Lambda} & \xleftarrow{y} & P^{-1}(C) \\
 & & \downarrow & & \downarrow & & \downarrow d_{P(C)}^{-1} \\
 & & \bar{\Lambda} & \xleftarrow{x^2} & \bar{\Lambda} & \xleftarrow{y} & P^0(C)
 \end{array}$$

We drew this schema by starting in degree zero as we read the word  $C$  from left to right.

For  $-\mathbb{N}$  words we read right to left, and for  $\mathbb{Z}$ -words we do the former then the latter.

**Isomorphisms.** Before we introduce complexes of the form  $P^\bullet(C, V)$  it is necessary to note that the complexes  $P^\bullet(C)$  and  $P^\bullet(C')$  are isomorphic given the words  $C$  and  $C'$  are related. Any word  $C$  defines a subset  $I_C \subseteq \mathbb{Z}$  for which  $C$  is an  $I_C$ -word. The following is by definition.

**Lemma 1.1.** Fix a word  $C$  and some  $i \in I_C$ .

- (i) If  $I_C = \{0, \dots, m\}$  then  $I_{C^{-1}} = I_C$ ,  $v_{C^{-1}}(i) = v_C(m-i)$  and  $\mu_{C^{-1}}(i) = \mu_C(m-i) - \mu_C(m)$ .
- (ii) If  $I_C$  is infinite then  $-I_C = \{-i \mid i \in I_C\}$ ,  $v_{C^{-1}}(i) = v_C(-i)$  and  $\mu_{C^{-1}}(i) = \mu_C(-i)$ .

Not all complexes of the form  $P^\bullet(C)$  have finitely generated homogeneous components.

**Example (GP).** For the algebra  $\bar{\Lambda} = k[[x, y]]/(xy)$  let  $D$  be the word

$$^\infty(d_y^{-1}yx^{-1}d_x) \mid (d_y^{-1}yx^{-1}d_x)^\infty = \dots x^{-1}d_x d_y^{-1}yx^{-1}d_x \mid d_y^{-1}yx^{-1}d_x d_y^{-1}y \dots =$$

gives a complex  $P^\bullet(D)$  whose homogeneous component of degree 1 is isomorphic to  $\bigoplus_{\mathbb{Z}} \bar{\Lambda}$ .  $D$  is an example of what we call a *periodic*  $\mathbb{Z}$ -word with period 2.

**Shift of a word.** Let us recall and adapt some terminology from [13]. For  $t \in \mathbb{Z}$  and a  $\mathbb{Z}$ -word  $C = \dots l_0^{-1}r_0 \mid l_1^{-1}r_1 \dots$  the *shift*  $C[t]$  of  $C$  by  $t$  will be the word  $\dots l_t^{-1}r_t \mid l_{t+1}^{-1}r_{t+1} \dots$ . We extend this definition to all  $I$ -words  $C$  where  $I \neq \mathbb{Z}$  by setting  $C = C[t]$  for all  $t \in \mathbb{Z}$ . In the case of a  $\mathbb{Z}$ -word we may book-keep as we did in lemma 1.1. The proof is again by definition.

**Lemma 1.2.** For any  $\mathbb{Z}$ -word  $C$  and any  $t \in \mathbb{Z}$  we have  $v_C(i+t) = v_{C[t]}(i)$  and  $\mu_C(i+t) = \mu_{C[t]}(i) + \mu_C(t)$  for each  $i \in \mathbb{Z}$ .

**Corollary 1.1.** Let  $C$  be an  $I$ -word.

- (i) If  $I = \{0, \dots, m\}$  there is an isomorphism of complexes  $P^\bullet(C^{-1}) \rightarrow P^\bullet(C)[\mu_C(m)]$ .
- (ii) If  $I$  is infinite there is an isomorphism of complexes  $P^\bullet(C^{-1}) \rightarrow P^\bullet(C)$ .
- (iii) If  $I = \mathbb{Z}$  there is an isomorphism of complexes  $P^\bullet(C[t]) \rightarrow P^\bullet(C)[\mu_C(t)]$  for each  $t \in \mathbb{Z}$ .

*Proof.* Fix  $n \in \mathbb{Z}$ . For each  $i \in \mu_C^{-1}(n)$  let  $b_{i,C^{-1}}$  denote the coset of  $e_{v_{C^{-1}}(i)}$  in the summand  $\bar{\Lambda}e_{v_{C^{-1}}(i)}$  of  $P^n(C^{-1})$ . Similarly for each  $j \in \mu_C^{-1}(n + \mu_C(m))$  let  $b'_{j,C}$  denote the coset of  $e_{v_C(j)}$  in the summand  $\bar{\Lambda}e_{v_C(j)}$  of  $P^{n+\mu_C(m)}(C)$ .

Extending the assignment  $\theta^n(b_{i,C^{-1}}) = b'_{m-i,C}$  linearly over  $\bar{\Lambda}$  gives a well defined module map by lemma 1.1 (parts (i) and (ii) resp.). Part (iii) of lemma 1.1 shows that  $\theta^n$  is a module isomorphism  $P^n(C^{-1}) \rightarrow P^n(C)[\mu_C(m)]$ , and by definition  $\theta^\bullet$  is a morphism of complexes. This gives (i). (ii) and (iii) are similar to the above. For (ii) the case where  $I$  is infinite in lemma 1.1 is used, where as for (iii) one applies lemma 1.2.  $\square$

If  $C[t] = C$  and  $\mu_C(t) = 0$  then this isomorphism from part (iii) is an automorphism.

**Example (GP).** When  $D = {}^\infty(d_y^{-1}yx^{-1}d_x) \mid (d_y^{-1}yx^{-1}d_x)^\infty$  the automorphisms  $t_D^0$  of  $P^0(D)$  and  $t_D^1$  of  $P^1(D)$  may be depicted as

$$\begin{array}{ccccccc} \dots & \xleftarrow{t_D^0} & \bar{\Lambda} & \xleftarrow{t_D^0} & \bar{\Lambda} & \xleftarrow{t_D^0} & \bar{\Lambda} & \xleftarrow{t_{D,2}^0} & \dots & P^0(D) \\ & \searrow & \swarrow x & \searrow & \swarrow x & \searrow & \swarrow x & \searrow & & \downarrow d_{P(D)}^0 \\ \dots & \xleftarrow{t_D^1} & \bar{\Lambda} & \xleftarrow{t_D^1} & \bar{\Lambda} & \xleftarrow{t_D^1} & \bar{\Lambda} & \xleftarrow{t_{D,2}^1} & \dots & P^1(D) \end{array}$$

**Complexes of the form  $P^\bullet(C, V)$ .** We say a word  $C$  is *periodic* if  $C = C[p]$  and  $\mu_C(p) = 0$  for some  $p > 0$  and the *period* of  $C$  describes the minimal such  $p$ . In this case there is an automorphism  $t_C^\bullet$  of  $P^\bullet(C)$  sending  $b_i$  to  $b_{i-p}$ . In this way  $P(C)$  is a left  $\bar{\Lambda} \otimes_k k[T, T^{-1}]$ -module by letting  $T$  act on the right by  $t_C$ . By lemma 1.2 we have  $\mu_C(i+p) = \mu_C(i)$  for each  $i \in \mathbb{Z}$  so  $t_C$  is of degree 0 which means  $P^n(C)$  is a left  $\bar{\Lambda} \otimes_k k[T, T^{-1}]$ -submodule of  $P(C)$  for each  $n \in \mathbb{Z}$ . Now fix a  $k[T, T^{-1}]$ -module  $V$ . By definition  $d_{P(C)}^n$  defines a homomorphism of  $\bar{\Lambda} \otimes_k k[T, T^{-1}]$  modules. Note  $\mu_C(i) = \mu_C(i+p)$  for all  $i \in \mathbb{Z}$ .

Define the complex  $P^\bullet(C, V)$  by letting  $P^n(C, V) = P^n(C) \otimes_{k[T, T^{-1}]} V$  and  $d_{P(C, V)}^n = d_{P(C)}^n \otimes 1_V$  for each degree  $n \in \mathbb{Z}$ . There is a  $\bar{\Lambda}$ -module isomorphism defined by

$$\kappa : P(C, V) \rightarrow \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i \in \mu_P^{-1}(n)} \bar{\Lambda}e_{v_C(i)} \otimes_k V, \quad \left( \sum_{i \in \mu_C^{-1}(n)} m_i \right) \otimes v \mapsto \sum_{i \in \mu_P^{-1}(n)} \left( \sum_{r \in \mathbb{Z}} m_{i+pr} T^{-r} \otimes T^r v \right)$$

for  $m_i \in \bar{\Lambda}e_{v_C(i)}$  and  $v \in V$ . Consequently  $P(C, V)$  is a projective  $\bar{\Lambda}$ -module generated by the elements  $b_j \otimes v_\lambda$  where  $j$  runs through  $\{0, \dots, p-1\}$  and  $\{v_\lambda \mid \lambda \in \Omega\}$  is a basis of  $V$ . So  $P^n(C, V)$  is finitely generated for each  $n \in \mathbb{Z}$  given  $\dim(V)$  is finite.

**Example (GP).** For example if  $V$  is the  $k[T, T^{-1}]$  module  $kv_0 \cong k$  where  $T$  acts as multiplication by a scalar  $\mu \neq 0$  then  $P^\bullet(D, V)$  is given by the schema

$$x \begin{pmatrix} \overline{\Lambda} \\ \mu \\ \overline{\Lambda} \end{pmatrix} y$$

where  $d_{P(D, V)}^0(b_0 \otimes v_0) = \mu x b_{-1} \otimes v_0 + y b_{-1} \otimes v_0$ .

**Example (ALP).** If  $k$  is algebraically closed, any finite dimensional indecomposable  $k[T, T^{-1}]$ -module  $V$  has the form the  $k^n$  where  $T$  acts as multiplication by a Jordan block  $J_n(\lambda)$ . Let  $C$  be the word  ${}^\infty E \mid E^\infty$  where  $E = d_t^{-1} t d_s^{-1} s d_r^{-1} r a^{-1} d_a b^{-1} d_b c^{-1} d_c$ .  $P^\bullet(C, V)$  is given by the schema

$$\begin{array}{ccccc} & \oplus_{i=1}^n \Gamma e_0 & & & \\ & \swarrow c & & \searrow t & \\ \oplus_{i=1}^n \Gamma e_2 & & J_n(\lambda) & & \oplus_{i=1}^n \Gamma e_3 \\ \downarrow b & & & & \downarrow s \\ \oplus_{i=1}^n \Gamma e_1 & & & & \oplus_{i=1}^n \Gamma e_4 \\ & \searrow a & & \swarrow r & \\ & \oplus_{i=1}^n \Gamma e_0 & & & \end{array}$$

where  $d_{P(C, V)}^0(b_0 \otimes v) = cb_5 \otimes J_n(\lambda)v + tb_1 \otimes v$ .

**Exchanging  $T$  and  $T^{-1}$ .** Let  $\iota$  define the  $k$ -algebra automorphism of  $k[T, T^{-1}]$  which exchanges  $T$  and  $T^{-1}$ . Define a functor  $\text{res}_\iota : k[T, T^{-1}] - \mathbf{Mod} \rightarrow k[T, T^{-1}] - \mathbf{Mod}$  by setting  $\text{res}_\iota(V)$  to have underlying vector space  $V$  (for any  $k[T, T^{-1}]$ -module  $V$ ) but where the action of  $T$  on  $v \in \text{res}_\iota(V)$  is defined by  $T.v = T^{-1}v$ .

Hence  $\iota$  is an involution (that is  $\text{res}_\iota \circ \text{res}_\iota \cong 1_{k[T, T^{-1}] - \mathbf{Mod}}$ ). Later we see that  $P^\bullet(C, V)$  and  $P^\bullet(C^{-1}, \text{res}_\iota(V))$  are isomorphic complexes.

**String and band complexes.** We call  $P^\bullet(C)$  a *string complex* provided  $C$  is not a periodic  $\mathbb{Z}$ -word. When  $C$  is a periodic  $\mathbb{Z}$ -word we call  $P^\bullet(C, V)$  a *band complex* provided  $V$  is a finite dimensional indecomposable  $k[T, T^{-1}]$ -module.

**Theorem 1.1.** *Every complex of finitely generated projectives with radical images in the homotopy category is isomorphic to a direct sum of string and band complexes. Any indecomposable object in  $\mathcal{K}(\overline{\Lambda} - \mathbf{proj})$  is isomorphic to a shift of a string complex with controlled homogeny, or a band complex. Furthermore, all shifts of string complexes and band complexes are indecomposable complexes in  $\mathcal{K}(\overline{\Lambda} - \mathbf{Proj})$ .*

The next theorem informs the reader how to construct isomorphism classes of indecomposables. This is an analogue of [33, p.21, Theorem].

**Theorem 1.2.** *Let  $C$  and  $D$  be words and let  $V$  and  $W$  be  $k[T, T^{-1}]$ -modules.*

(I) *If  $C$  and  $D$  have controlled homogeny,  $P^\bullet(C) \cong P^\bullet(D)[m]$  iff one of the following hold:*

- (a)  *$C$  and  $D$  are  $\{0, \dots, t\}$ -words and  $(D = C \text{ and } m = 0)$  or  $(D = C^{-1} \text{ and } m = \mu_C(t))$ ,*
- (b)  *$C$  and  $D$  are  $(\mathbb{N} \text{ or } -\mathbb{N})$ -words,  $D = C^{\pm 1}$  and  $m = 0$ , or*
- (c)  *$C$  and  $D$  are  $\mathbb{Z}$ -words,  $D = C^{\pm 1}[t]$  and  $m = \mu_C(\pm t)$ .*

(II) *If  $C$  and  $D$  are periodic then  $P^\bullet(C, V) \cong P^\bullet(D, W)[m]$  iff one of the following hold:*

- (d)  *$D = C[t]$ ,  $V \cong W$  and  $m = \mu_C(t)$ , or*
- (e)  *$D = C^{-1}[t]$ ,  $V \cong \text{res}_\iota W$  and  $m = \mu_C(-t)$ .*

(III) *No isomorphisms exist between a shift of a string complex and a shift of a band complex.*

**Theorem 1.3.** *(Krull-Remak-Schmidt property). If a complex of finitely generated projectives with radical images is written as a direct sum of string and band complexes in two different ways there is a bijection between the summands in such a way that corresponding summands are isomorphic.*

## 2. LINEAR RELATIONS AND ONE-SIDED FUNCTORS.

In [13, Section 7] certain functors were evaluated on string and band modules and this calculation was essential in the final proof. These functors were defined as quotients of subfunctors of the forgetful functor (sending a module to its underlying vector space), and these subfunctors were constructed using the language of linear relations. To adapt the technique of functorial filtration to our setting we use a similar approach.

**Linear relations.** For vector spaces  $V$  and  $W$  a *linear relation*  $C$  from  $V$  to  $W$  is a subspace of  $V \oplus W$ . From now on relation means linear relation. The *converse*  $C^{-1} = \{(w, v) \mid (v, w) \in C\}$  of  $C$  defines a relation from  $W$  to  $V$ . For relations  $C$  from  $V$  to  $W$  and  $D$  from  $U$  to  $V$  the *composition*  $CD$  is the relation from  $U$  to  $W$  consisting of all  $(u, w) \in U \oplus W$  such that  $w \in Cv$  and  $v \in Du$  for some  $v \in V$ . For any  $t \in V$  the *image of  $C$  at  $t$*  is  $Ct = \{w \in W \mid (t, w) \in C\}$  and for a subset  $T \subseteq V$  we let  $CT = \bigcup_{t \in T} Ct$ . For complexes  $X^\bullet$  and  $Y^\bullet$  a *linear relation* from  $X^\bullet$  to  $Y^\bullet$  is a relation from  $X = \bigoplus_{i \in \mathbb{Z}} X^i$  to  $Y = \bigoplus_{i \in \mathbb{Z}} Y^i$ . A *linear relation on  $X^\bullet$*  is a linear relation from  $X$  to itself.

**Relations given by path multiplication.**  $M^\bullet$  will be a complex of (not necessarily finitely generated) projective  $\bar{\Lambda}$ -modules with radical images.  $M$  is the  $\bar{\Lambda}$ -module  $\bigoplus_{i \in \mathbb{Z}} M^i$  and  $d_M$  denotes the endomorphism  $\bigoplus_{i \in \mathbb{Z}} d_M^i$  of  $M$  sending  $\sum m_i$  to  $\sum d_M^i(m_i)$ . For  $\gamma \in \mathbf{Pa}_\rho$  consider the linear map  $\gamma_\times : e_{t(\gamma)}M \rightarrow e_{h(\gamma)}M$  sending  $\lambda$  to  $\gamma\lambda$ . Let  $(\gamma)$  denote relation defined by the graph of  $\gamma_\times$ , and write  $(\gamma^{-1})$  for the converse  $(\gamma)^{-1}$  of  $(\gamma)$ . For a subspace  $U$  of  $M$  one may consider the subspaces  $\gamma U$  and  $\gamma^{-1}U$  defined by

$$\gamma U = \{\gamma m \in e_{h(\gamma)}M \mid m \in U\}, \quad \gamma^{-1}U = \{m \in e_{t(\gamma)}M \mid \gamma m \in U\}.$$

We start by recording some consequences of dealing with projective modules over gentle algebras.

**Lemma 2.1.** *Fix arrows  $a$  and  $b$ .*

- (i) *If  $h(b) = t(a)$  and  $ab \notin \rho$ ,  $abm = 0$  implies  $bm = 0$  for any element  $m$  in  $M$ .*
- (ii)  *$\{m' \in e_{t(a)}M \mid am' = 0\} = \sum b'M$  where the sum ranges over all arrows  $b'$  with  $ab' \in \rho$ .*
- (iii) *Given an arrow  $b'$  where  $b' \neq b$  and  $h(b') = h(b)$  we have  $bM \cap b'M = 0$ .*

*Proof.* Since  $M$  is projective there is a split embedding  $\psi$  from  $M$  into a free module. This gives (i). Part (ii) is essentially [5, p. 299, Lemma 5]. For (iii) let  $S_b$  (resp.  $S_{b'}$ ) be the set of all  $b\sigma$  where  $\sigma$  (resp.  $\sigma'$ ) runs through the set of paths which satisfy  $b\sigma \in \mathbf{Pa}_\rho$  (resp.  $b'\sigma' \in \mathbf{Pa}_\rho$ ). Since  $b \neq b'$  the union of  $S_b$  and  $S_{b'}$  is disjoint and forms a linearly independent subset of  $\bar{\Lambda}$ . Using this any element  $y = bm = b'm'$  must be sent to 0 under  $\psi$ .  $\square$

We now give some meaning to the letters  $d_\alpha^{\pm 1}$  for arrows  $\alpha$ .

**The  $d$  maps.** The following should shed light on why we restricted our focus from  $\mathcal{K}(\bar{\Lambda} - \mathbf{Proj})$  to  $\mathcal{K}_{\text{rad}}(\bar{\Lambda} - \mathbf{Proj})$ .

**Lemma 2.2.** *For each arrow  $\alpha$  there is an endomorphism  $d_{\alpha, M}$  of the vector space  $e_{h(\alpha)}M$  such that for any vertex  $v$  the restriction  $d_M|_v$  of  $d_M$  to  $e_vM$  is the sum  $\sum_\beta d_{\beta, M}$  running over all arrows  $\beta$  with head  $v$ . Furthermore for any  $\tau \in \mathbf{Pa}_\rho$  and any  $x \in e_{t(\tau)}M$ ;*

- (i) *If  $\tau\sigma \in \mathbf{Pa}_\rho$  for  $\sigma \in \mathbf{Pa}_\rho$  then  $d_{f(\tau), M}(\tau x) = \tau d_{f(\sigma), M}(x)$  and otherwise  $d_{f(\tau), M}(\tau x) = 0$ .*
- (ii) *If  $h(\theta) = h(\tau)$  and  $f(\theta) \neq f(\tau)$  for some  $\theta \in \mathbf{Pa}_\rho$  we have  $d_{f(\theta), M}(\tau x) = 0$ .*
- (iii) *If  $h(\theta) = h(\tau)$  for  $\theta \in \mathbf{Pa}_\rho$  then  $d_{f(\theta), M}d_{f(\tau), M} = 0$ . In particular  $d_{f(\tau), M}^2 = 0$ .*
- (iv) *If  $\tau\sigma \in \mathbf{Pa}_\rho$  for  $\sigma \in \mathbf{Pa}_\rho$  we have  $d_{f(\sigma), M}(x) = 0$  given  $\tau x \in \text{im}(d_{f(\tau), M})$ .*

*Proof.* Since  $M^\bullet$  is a complex with radical images we have that  $\text{im}(d_M) \subseteq \bar{A}M$  (where  $A$  is the ideal of  $\Lambda$  generated by the arrows). Hence the image of  $d_M$  upon restriction to  $e_vM$  is contained in  $e_v\bar{A}M = \sum \beta M$  where  $\beta$  runs all arrows with head  $v$ .

So this sum runs over a set with at most 2 elements and is direct by lemma 2.1. For any arrow  $\gamma$  with head  $v$  let  $\pi_\gamma : \bigoplus \beta M \rightarrow \gamma M$  and  $\iota_\gamma : \gamma M \rightarrow \bigoplus \beta M$  be the natural maps.



Define  $d_{\alpha,M}$  to be the map sending  $m \in e_v M$  to  $\iota_\alpha(\pi_\alpha(d_M|_v(m)))$  which is an element of  $\bigoplus \beta M \subseteq e_v M$ . Then we have

$$\sum_{\beta} d_{\beta,M}(m) = \sum_{\beta} \iota_\beta(\pi_\beta(d_M|_v(m))) = \left( \sum_{\beta} \iota_\beta \pi_\beta \right) (d_M|_v(m)) = d_M|_v(m) \quad (\star_v).$$

Let  $v = h(\tau)$  and  $u = t(\tau)$ . By definition  $d_{f(\tau),M}(\tau x) = \iota_{f(\tau)}(\pi_{f(\tau)}(d_M(\tau x))) = \tau d_M|_u(x)$ . For any  $\theta \in \mathbf{Pa}_\rho$  with head  $u$  and  $f(\theta) \neq f(\sigma)$  we have  $l(\tau)f(\theta) \in \rho$  since  $\bar{\Lambda}$  is a gentle algebra. The equations  $\tau d_M|_{e_u M}(x) = d_{f(\tau),M}(\tau x)$  and  $(\star_u)$  together give (i). (ii) is clear as  $\pi_{f(\theta)}(\tau x) = 0$  by definition. (iii) and (iv) follow by the above and lemma 2.1.  $\square$

**Relations given by words.** Let  $\alpha$  be an arrow. We start by explaining how relations may be defined by the maps  $d_{\alpha,M} : e_{h(\alpha)} M \rightarrow e_{h(\alpha)} M$  from the above lemma. Let  $(d_\alpha, M)$  denote relation defined by the graph of  $d_{\alpha,M}$  and as above let  $(d_\alpha^{-1}, M) = (d_\alpha, M)^{-1}$ . For each letter  $q$  the image  $qU$  of the relation  $(q, U)$  is given by

$$\begin{aligned} \gamma U &= \{\gamma m \in e_{h(\gamma)} M \mid m \in U\}, & \gamma^{-1} U &= \{m \in e_{t(\gamma)} M \mid \gamma m \in U\}, \\ d_\alpha U &= \{d_{\alpha,M}(m) \in e_{h(\alpha)} M \mid m \in U\}, & d_\alpha^{-1} U &= \{m \in e_{h(\alpha)} M \mid d_{\alpha,M}(m) \in U\}. \end{aligned}$$

For any non-trivial finite  $I = \{0, \dots, m\}$ -word  $C = l_1^{-1} r_1 \dots l_m^{-1} r_m$  and any  $i \in \{1, \dots, m\}$ , the linear relation  $(l_i^{-1} r_i, M)$  from  $e_{v_C(i-1)} M$  to  $e_{v_C(i)} M$  consists of all pairs  $(m_{i-1}, m_i)$  where

$$\gamma m_{i-1} = d_{f(\gamma),M}(m_i) \text{ if } l_i^{-1} r_i = \gamma^{-1} d_{f(\gamma)}, \text{ and } d_{f(\gamma),M}(m_{i-1}) = \gamma m_i \text{ if } l_i^{-1} r_i = d_{f(\gamma)}^{-1} \gamma.$$

For each subspace  $U$  of  $e_{h(C-1)} M$  we let  $CU = \bigcup_{x \in U} Cx$  where  $Cx$  is the set of all  $x_0 \in e_{h(C)} M$  such that there are elements  $x_i \in e_{v_C(i)} M$  where  $x = x_m$  and  $(x_{i-1}, x_i) \in (l_i^{-1} r_i, M)$  for  $1 \leq i \leq m$ . In the cases  $U = e_{h(C-1)} M$ ,  $U = e_{h(C-1)} \text{rad}(M)$  and  $U = \{0\}$  we write  $CM$ ,  $C\text{rad}(M)$ , and  $C0$  respectively. For any subspace  $W$  of  $e_v M$ , each trivial word  $1_{v,\pm 1}$  defines a relation  $(1_{v,\pm 1}, W)$  consisting of pairs  $(m, m)$  with  $m \in W$ . By lemma 2.2 the assignment  $M^\bullet \mapsto CM$  defines a subfunctor of the forgetful functor  $\mathcal{C}_{\text{rad}}(\bar{\Lambda} - \mathbf{Proj}) \rightarrow k - \mathbf{Mod}$  taking a complex  $M^\bullet$  to its underlying vector space  $\bigoplus_{i \in \mathbb{Z}} M^i$ .

**Corollary 2.1.** *If  $a$  is an arrow then  $a^{-1} d_a \text{rad}(M) \subseteq e_{t(a)} \text{rad}(M)$ . Furthermore, given an arrow  $b$  with  $ab \in \mathbf{Pa}_\rho$  we have  $(ab)^{-1} a d_b M = b^{-1} d_b M$ .*

*Proof.* Follows from lemmas 2.1 and 2.2.  $\square$

We now gather some consequences of lemma 2.2 in the language of relations recalled above. The next corollary follows from lemma 2.2.

**Corollary 2.2.** *Let  $\alpha, \beta, \gamma$  and  $\sigma$  be paths in  $\mathbf{Pa}_\rho$  with  $\alpha\beta \in \mathbf{Pa}_\rho$ ,  $h(\gamma) = h(\sigma)$  and  $f(\gamma) \neq f(\sigma)$ . Then we have the inclusions*

$$\begin{aligned} \beta^{-1} d_{f(\beta)} M &\subseteq (\alpha\beta)^{-1} d_{f(\alpha)} M, & d_{f(\alpha)}^{-1} \alpha\beta M &\subseteq d_{f(\alpha)}^{-1} \alpha M, \\ \alpha^{-1} d_{f(\alpha)} M &\subseteq d_{f(\beta)}^{-1} \beta 0, & \gamma M &\subseteq d_{f(\sigma)}^{-1} \sigma 0, & \text{and } d_{f(\sigma)} M &\subseteq d_{f(\sigma)}^{-1} \sigma 0. \end{aligned}$$

We look toward introducing functors  $C^\pm$  defined by a word  $C$ , adapting notions used by Ringel [33, p. 23] and Crawley-Boevey [13, p. 11]. At this point it is necessary to describe how words are composed, adapted from [13] for our purposes. Let  $Q_1^{\pm}$  be the set of letters  $\alpha$  or  $\alpha^{-1}$  where  $\alpha$  is an arrow.

**Sign convention.** We choose a *sign*  $s(q) \in \{\pm 1\}$  for each letter  $q$  in  $Q_1^\pm$ , such that if distinct letters  $q$  and  $q'$  from  $Q_1^\pm$  have the same head, they have the same sign only if  $\{q, q'\} = \{\alpha^{-1}, \beta\}$  with  $\alpha\beta \in \rho$ . We extend this notion to all letters by letting  $s(\gamma) = s(f(\gamma))$ ,  $s(\gamma^{-1}) = s(l(\gamma)^{-1})$ , and  $s(d_\alpha^{\pm 1}) = -s(\alpha)$  for each  $\gamma \in \mathbf{Pa}_\rho$  and each arrow  $\alpha$ . For a (non-trivial finite or  $\mathbb{N}$ )-word  $C$  we let  $h(C)$  and  $s(C)$  be the head and sign of the first letter of  $C$ . For the trivial words  $1_{v,\pm 1}$  we let  $s(1_{v,\pm 1}) = \pm 1$  and  $h(1_{v,\pm 1}) = v$ .

**Word composition.** We may *compose* non-trivial words  $C$  and  $D$  to form the *composition*  $CD$  by concatenating the letters, provided;  $I_D \subseteq \mathbb{N}$ ,  $I_C \subseteq -\mathbb{N}$ ,  $h(C^{-1}) = h(D)$  and  $s(C^{-1}) = -s(D)$ . The result is a word. When we may compose the words  $C$  and  $D$  we just say  $CD$  is a word. Given they are composable (i.e. for appropriate  $\delta, \epsilon \in \{\pm 1\}$ ) we stipulate that  $1_{v,\delta}C = C$  and  $C1_{u,\epsilon} = C$ . The composition of a  $-\mathbb{N}$ -word  $C = \dots l_{-1}^{-1}r_{-1}l_0^{-1}r_0$  and an  $\mathbb{N}$ -word  $D = l_1^{-1}r_1l_2^{-1}r_2\dots$  is a  $\mathbb{Z}$ -word indexed so that  $CD = \dots l_0^{-1}r_0 \mid l_1^{-1}r_1\dots$ . The following is obvious.

**Lemma 2.3.** *Suppose  $C$  is a finite word. If both  $C\gamma^{-1}d_{f(\gamma)}$  and  $C\beta^{-1}d_{f(\beta)}$  (resp.  $Cd_{f(\gamma)}^{-1}\gamma$  and  $Cd_{f(\alpha)}^{-1}\alpha$ ) are words then  $l(\gamma) = l(\beta)$  (resp.  $f(\gamma) = f(\alpha)$ ) and consequently  $\gamma = \alpha\beta$  for some  $\alpha$  (resp.  $\beta$ ) if  $\gamma$  is longer than  $\beta$  (resp.  $\alpha$ ).*

**Corollary 2.3.** *Let  $C$  be a finite word and suppose  $a$  and  $b$  are arrows such that  $Ca^{-1}d_a$  and  $Cd_b^{-1}b$  are words. Then:*

- (i)  $C\gamma^{-1}d_{f(\gamma)}$  (resp.  $Cd_{f(\tau)}^{-1}\tau$ ) is a word if and only if  $l(\gamma) = a$  (resp.  $f(\tau) = b$ ),
- (ii) for words  $C\gamma^{-1}d_{f(\gamma)}$  and  $C\gamma'^{-1}d_{f(\gamma')}$ , if  $\gamma'$  is longer than  $\gamma$  then  $C\gamma^{-1}d_{f(\gamma)}M \subseteq C\gamma'^{-1}d_{f(\gamma')}M$ ,
- (iii) for words  $Cd_{f(\tau)}^{-1}\tau$  and  $Cd_{f(\tau')}^{-1}\tau'$ , if  $\tau'$  is longer than  $\tau$  then  $Cd_{f(\tau')}^{-1}\tau'M \subseteq Cd_{f(\tau)}^{-1}\tau M$ .

*Proof.* Follows by corollary 2.2 and lemma 2.3.  $\square$

**Example.** For the gentle algebra  $k[[x, y]]/(xy)$  we have

$$M = d_x^{-1}xM \supseteq d_x^{-1}x^2M \supseteq d_x^{-1}x^3M \supseteq \dots \supseteq x^{-3}d_xM \supseteq x^{-2}d_xM \supseteq x^{-1}d_xM$$

We can now define the sub-spaces of  $M$  which will be the building blocks of our refined functors. If  $C = \dots l_i^{-1}r_i\dots$  is a word and  $i \in I_C$  is arbitrary; we let  $C_i = l_i^{-1}r_i$  and  $C_{\leq i} = \dots l_i^{-1}r_i$  given  $i-1 \in I_C$ , and otherwise  $C_i = C_{\leq i} = 1_{h(C), s(C)}$ . Similarly we let  $C_{> i} = l_{i+1}^{-1}r_{i+1}\dots$  given  $i+1 \in I_C$  and otherwise  $C_{> i} = 1_{h(C^{-1}), s(C^{-1})}$ . Hence there are unique words  $C_{< i}$  and  $C_{\geq i}$  satisfying  $C_{\leq i} = C_{< i}C_i$  and  $C_iC_{> i} = C_{\geq i}$ .

**One-sided  $C^\pm$  functors.** For each vertex  $v$  and sign  $\delta \in \{\pm 1\}$  let  $\mathcal{W}_{v,\delta}$  be the set of all (finite or  $\mathbb{N}$ )-words with head  $v$  and sign  $\delta$ . Suppose  $C$  is a finite word from  $\mathcal{W}_{v,\delta}$ .

We let  $C^+(M^\bullet) = \bigcap C d_{f(\gamma)}^{-1} \gamma \text{rad}(M)$  where the intersection is taken over all  $\gamma$  for which  $C d_{f(\gamma)}^{-1} \gamma$  is a word provided one exists, and let  $C^+(M^\bullet) = CM$  otherwise.

Let  $C^-(M^\bullet) = \bigcup C \beta^{-1} d_{f(\beta)} M$  where the union is taken over all  $\beta$  for which  $C \beta^{-1} d_{f(\beta)}$  is a word, given at least one such  $\beta$  exists. Otherwise let  $C^-(M^\bullet) = C(\sum_{\alpha_-} d_{\alpha_-} M + \sum_{\alpha_+} \alpha_+ M)$  where  $\alpha_\pm$  runs through all arrows with head  $h(C^{-1})$  and sign  $\pm s(C^{-1})$ .

Suppose now  $C = l_1^{-1}r_1 \dots l_i^{-1}r_i \dots$  is an  $\mathbb{N}$ -word from  $\mathcal{W}_{v,\delta}$ . In this case let  $C^+(M^\bullet)$  be the set of all  $m_0 \in e_v M$  with a sequence of elements  $(m_i)_{i \in \mathbb{N}} \in \prod e_{v_C(i)} M$  satisfying  $m_i \in l_{i+1}^{-1}r_{i+1}m_{i+1}$  for each  $i \in \mathbb{N}$ , and let  $C^-(M^\bullet)$  be the subset of  $C^+(M^\bullet)$  where each sequence  $(m_i)_{i \in \mathbb{N}}$  is eventually zero. Equivalently  $C^-(M^\bullet) = \bigcup_{n \in \mathbb{N}} C_{\leq n} 0$ .

### 3. ORDERING WORDS AND INTERVAL AVOIDANCE.

Fix some vertex  $v$  and some  $\delta \in \{\pm 1\}$ . We now introduce an ordering on the set  $\mathcal{W}_{v,\delta}$  of all (finite or  $\mathbb{N}$ )-words with head  $v$  and sign  $\delta$ . To do so an ordering is introduced on the set of pairs  $(l, r)$  of letters for which a word  $C$  may be extended to a word  $Cl^{-1}r$ . The following is immediate by lemma 2.3.

**Lemma 3.1.** *Suppose  $l, l', r$ , and  $r'$  are letters for which  $l^{-1}r$  and  $l'^{-1}r'$  are distinct words in  $\mathcal{W}_{v,\delta}$ . Then there exists distinct elements  $\alpha$  and  $\alpha'$  from  $\mathbf{Pa}_\rho$  such that one of the following hold:*

- (i)  $l^{-1}r = d_{f(\alpha)}^{-1}\alpha$  and  $l'^{-1}r' = d_{f(\alpha')}^{-1}\alpha'$  where  $f(\alpha) = f(\alpha')$ ,
- (ii)  $l^{-1}r = d_{f(\alpha)}^{-1}\alpha$  and  $l'^{-1}r' = \alpha'^{-1}d_{f(\alpha')}$  where  $\alpha'\alpha \in \mathbf{Pa}_\rho$ ,
- (iii)  $l^{-1}r = \alpha^{-1}d_{f(\alpha)}$  and  $l'^{-1}r' = \alpha'^{-1}d_{f(\alpha')}$  where  $l(\alpha) = l(\alpha')$ , or
- (iv)  $l^{-1}r = \alpha^{-1}d_{f(\alpha)}$  and  $l'^{-1}r' = d_{f(\alpha')}^{-1}\alpha'$  where  $\alpha\alpha' \in \mathbf{Pa}_\rho$ .

**Definition 3.1.** If  $l^{-1}r$  and  $l'^{-1}r'$  are distinct words in  $\mathcal{W}_{v,\delta}$  we say  $l^{-1}r < l'^{-1}r'$  if one of the following hold:

- (I)  $l^{-1}r = d_{f(\gamma)}^{-1}\gamma$  and  $l'^{-1}r' = d_{f(\gamma)}^{-1}\gamma\nu$  for some  $\gamma, \nu \in \mathbf{Pa}_\rho$  such that  $\gamma\nu \in \mathbf{Pa}_\rho$ ,
- (II)  $l^{-1}r = \mu^{-1}d_{f(\mu)}$  and  $l'^{-1}r' = d_{f(\eta)}^{-1}\eta$  for some  $\mu, \eta \in \mathbf{Pa}_\rho$  such that  $l(\mu)f(\eta) \in \mathbf{Pa}_\rho$ ,
- (III)  $l^{-1}r = \lambda^{-1}d_{f(\lambda)}$  and  $l'^{-1}r' = \lambda^{-1}\kappa^{-1}d_{f(\kappa)}$  for some  $\kappa, \lambda \in \mathbf{Pa}_\rho$  such that  $\kappa\lambda \in \mathbf{Pa}_\rho$ .

Since  $\bar{\Lambda}$  is gentle  $\mu\eta \in \mathbf{Pa}_\rho$  if and only if  $l(\mu)f(\eta) \in \mathbf{Pa}_\rho$  for any  $\mu, \eta \in \mathbf{Pa}_\rho$ . Together with lemma 3.1 this shows distinct pairs  $l^{-1}r$  and  $l'^{-1}r'$  from  $\mathcal{W}_{v,\delta}$  are comparable. Transitivity follows by case analysis. Consequently the relation  $<$  from the above definition gives a total order on the set of 2 letter words  $l^{-1}r \in \mathcal{W}_{v,\delta}$ .

**Example.** For the algebra  $k[[x, y]]/(xy)$  we have

$$d_x^{-1}x > d_x^{-1}x^2 > d_x^{-1}x^3 > \dots > x^{-3}d_x > x^{-2}d_x > x^{-1}d_x$$

Our construction ensures we can extend this lexicographically to a total order on  $\mathcal{W}_{v,\delta}$  as follows.

**Definition 3.2.** For distinct words  $C, C'$  from  $\mathcal{W}_{v,\delta}$  we say  $C < C'$  if one of the following hold:

- (I) there are letters  $l, l', r$  and  $r'$  and words  $B, D, D'$  for which  $C = Bl^{-1}rD$ ,  $C' = Bl'^{-1}r'D'$  and  $l^{-1}r < l'^{-1}r'$
- (II) there is some  $\beta \in \mathbf{Pa}_\rho$  for which  $C' = Cd_{f(\beta)}^{-1}\beta E$  for some word  $E$ ,
- (III) there is some  $\alpha \in \mathbf{Pa}_\rho$  for which  $C = C'\alpha^{-1}d_{f(\alpha)}E'$  for some word  $E'$ .

For the algebra  $\bar{\Lambda} = k[[x, y]]/(xy)$  we have for instance

$$d_x^{-1}xd_y^{-1}y^2 > d_x^{-1}x > d_x^{-1}xy^{-1}d_y > d_x^{-1}x^2 > d_x^{-1}x^3 > \dots > x^{-3}d_x > x^{-2}d_x > x^{-1}d_x.$$

We now use this ordering to adapt the interval avoidance lemma, as described in [33]. The next two lemmas are key in providing several key properties about our functors  $C^\pm$ . For example we shall see that such functors can be linearly ordered by the ordering placed words, and the refined functors (introduced in the next section) are well defined on the homotopy category. In what follows  $M^\bullet$  is an arbitrary complex of projective  $\bar{\Lambda}$ -modules with radical images. The following is straightforward.

**Lemma 3.2.** Suppose  $C$  is a finite word from  $\mathcal{W}_{v,\delta}$  and that there exists some  $\gamma \in \mathbf{Pa}_\rho$  for which  $Cd_{f(\gamma)}^{-1}\gamma$  is a word.

- (i) If infinitely many such  $\gamma$  exist then  $C^+(M^\bullet) = \bigcap_\gamma Cd_{f(\gamma)}^{-1}\gamma M$ .
- (ii) If only finitely many such  $\gamma$  exist then  $C^+(M^\bullet) = Cd_{f(\gamma)}^{-1}0$ .

The next result follows as an application of the above and corollary 2.2.

**Lemma 3.3.** For each  $i \in \mathcal{J}$  suppose  $X_i^\bullet$  is a complex of projectives with underlying graded vector space  $X_i$ . Then we have  $C^\pm(\bigoplus_{i \in \mathcal{J}} X_i^\bullet) = \bigoplus_{i \in \mathcal{J}} C^\pm(X_i^\bullet)$  for any finite or  $\mathbb{N}$ -word  $C$ .

In section 4 we introduce what are called *refined functors*, and the next lemma will be the key to showing refined functors are well defined on the homotopy category.

**Lemma 3.4.** For  $\alpha \in \mathbf{Pa}_\rho$  and any word  $C \in \mathcal{W}_{h(\alpha), s(\alpha)}$  we have  $f(\alpha)M \subseteq C^-(M^\bullet)$ .

*Proof.* By definition  $s(C) = s(\alpha)$  and  $h(w) = h(\alpha)$ . Suppose firstly that  $C$  is trivial so that  $C = 1_{h(\alpha), s(\alpha)}$ . If there is some  $\beta \in \mathbf{Pa}_\rho$  for which  $C\beta^{-1}d_{f(\beta)}$  is a word then  $f(\alpha)M \subseteq \beta^{-1}0$  which is contained in  $C^-(M^\bullet)$ . Otherwise there is no  $\beta \in \mathbf{Pa}_\rho$  for which  $C\beta^{-1}d_{f(\beta)}$  is a word which means  $f(\alpha)M \subseteq \sum_\gamma \gamma M$  where  $\gamma$  runs through all arrows with head  $h(C^{-1})$  and sign  $s(C^{-1})$ , and so  $f(\gamma)M \subseteq C^-(M^\bullet)$ .

Now assume  $C$  is non-trivial. If  $C = \beta^{-1}d_{f(\beta)}D$  for some word  $D$  then as before  $l(\beta)f(\alpha) \in \rho$  and again  $f(\alpha)M \subseteq \beta^{-1}0 \subseteq C^-(M^\bullet)$ . The last possibility is that  $C = d_{f(\gamma)}^{-1}\gamma E$  for some word  $E$  and some  $\gamma \in \mathbf{Pa}_\rho$ . Here  $f(\alpha)M \subseteq d_{f(\gamma)}^{-1}0$  by lemma 2.2 (i) as required.  $\square$

**Proposition 3.1.** (*Interval avoidance*) For any complex  $M^\bullet$  of projectives with radical images, if  $C$  and  $C'$  in  $\mathcal{W}_{v,\delta}$  with  $C < C'$  then  $C^+(M^\bullet) \subseteq C'^-(M^\bullet)$ .

*Proof.* For the case of (finite) words (for finite dimensional modules over  $k\langle x, y \rangle / (x^2, y^2)$ ) see [33, p 23, Lemma]. Fix words  $B$ ,  $D$ , and  $D'$ , and letters  $l$ ,  $l'$ ,  $r$  and  $r'$  such that  $Bl^{-1}rD$  and  $Bl'^{-1}r'D'$  are words in  $\mathcal{W}_{v,\delta}$ . If  $l^{-1}r < l'^{-1}r'$  then we have  $(l^{-1}rD)^+(M^\bullet) \subseteq (l'^{-1}r'D')^-(M^\bullet)$  by lemmas 2.2 and 3.4. The claim then follows by lemmas 3.2.  $\square$

**Example.** To put this account of functorial filtration into context, we now draw a diagram reminiscent of the depiction of a Cantor set. The purpose of this diagram is to illustrate to the reader the filtration of subspaces that may be defined (on a complex of projectives with radical images), and where the one sided functors may be seen.

$$\begin{array}{ccc}
M = d_y^{-1}yM & & d_y^{-1}yyP = d_y^{-1}yyd_x^{-1}xM \\
\vdots & \dashv & \vdots \\
d_y^{-1}yyM & & d_y^{-1}yyd_x^{-1}xxM \\
\vdots & & \vdots \\
d_y^{-1}yyyM & & d_y^{-1}yyd_x^{-1}xxxM \\
\vdots & & \vdots \\
(1_{1,1})^+(M^\bullet) = \bigcap d_y^{-1}yy \dots yM & & (d_y^{-1}yy)^+(M^\bullet) = d_y^{-1}yy \bigcap d_x^{-1}xx \dots xM \\
(1_{1,1})^-(M^\bullet) = \bigcup y^{-1}y^{-1} \dots y^{-1}d_yM & & (d_y^{-1}yy)^-(M^\bullet) = d_y^{-1}yy \bigcup x^{-1}x^{-1} \dots x^{-1}d_xM \\
\vdots & & \vdots \\
y^{-1}y^{-1}d_yM & & d_y^{-1}yyx^{-1}x^{-1}d_xM \\
\vdots & & \vdots \\
y^{-1}d_yM & & d_y^{-1}yyx^{-1}d_xM \\
\vdots & & \vdots \\
xM + d_yM & & d_y^{-1}yyyM = d_y^{-1}yy(yM + d_xM)
\end{array}$$

#### 4. FUNCTORS.

In what follows  $M^\bullet$  and  $N^\bullet$  will denote arbitrary complexes of projectives with radical images. **Refined functors.** At the end of section 2 we gave a way of defining subspaces  $C^-(M^\bullet) \subseteq C^+(M^\bullet) \subseteq e_v M$  for any given word  $C$  with head  $v$ . For each  $n \in \mathbb{Z}$  and each pair  $(B, D)$  of words with head  $v$  such that  $B^{-1}D$  is a word, define subspaces  $F_{B,D,n}^-(M^\bullet) \subseteq F_{B,D,n}^+(M^\bullet) \subseteq e_v M^n$  and  $G_{B,D,n}^-(M^\bullet) \subseteq G_{B,D,n}^+(M^\bullet) \subseteq e_v M^n$  by

$$\begin{aligned}
F_{B,D,n}^+(M^\bullet) &= M^n \cap (B^+(M^\bullet) \cap D^+(M^\bullet)), \\
F_{B,D,n}^-(M^\bullet) &= M^n \cap (B^-(M^\bullet) \cap D^-(M^\bullet) + B^-(M^\bullet) \cap D^+(M^\bullet)), \\
G_{B,D,n}^+(M^\bullet) &= M^n \cap (B^-(M^\bullet) + D^+(M^\bullet) \cap B^+(M^\bullet)), \\
G_{B,D,n}^-(M^\bullet) &= M^n \cap (B^-(M^\bullet) + D^-(M^\bullet) \cap B^+(M^\bullet)).
\end{aligned}$$

Now consider the quotients  $F_{B,D,n}(M^\bullet) = F_{B,D,n}^+(M^\bullet)/F_{B,D,n}^-(M^\bullet)$  and  $G_{B,D,n}(M^\bullet) = G_{B,D,n}^+(M^\bullet)/G_{B,D,n}^-(M^\bullet)$ . Let  $k[T, T^{-1}] - \mathbf{f.d.mod}$  be the full subcategory of  $k[T, T^{-1}] - \mathbf{Mod}$  consisting of finite dimensional modules. In this section our first goal is to prove the following.

**Proposition 4.1.** *In the above notation:*

- (a)  $F_{B,D,n}$  and  $G_{B,D,n}$  define naturally isomorphic functors from  $\mathcal{K}_{rad}(\overline{\Lambda} - \mathbf{Proj})$  to  $k - \mathbf{Mod}$ ,
- (b) if  $B^{-1}D$  is a periodic  $\mathbb{Z}$ -word,  $F_{B,D,n}$  and  $G_{B,D,n}$  give functors to  $k[T, T^{-1}] - \mathbf{Mod}$ ,
- (c) the restriction of  $F_{B,D,n}$  to  $\mathcal{K}_{rad}(\overline{\Lambda} - \mathbf{proj})$  has codomain  $k - \mathbf{mod}$ ,
- (d) if  $B^{-1}D$  is a periodic  $\mathbb{Z}$ -word, the restriction of  $F_{B,D,n}$  and  $G_{B,D,n}$  to  $\mathcal{K}_{rad}(\overline{\Lambda} - \mathbf{proj})$  has codomain  $k[T, T^{-1}] - \mathbf{f.d.mod}$ .

By a *refined functor* we will mean  $F_{B,D,n}$  or  $G_{B,D,n}$  for some  $n \in \mathbb{Z}$  and some words  $B$  and  $D$  for which  $B^{-1}D$  is a word.

**Corollary 4.1.** *For any  $\gamma \in \mathbf{Pa}_\rho$  and any  $n \in \mathbb{Z}$  we have  $M^n \cap \gamma^{-1} d_{f(\gamma)} M = M^n \cap \gamma^{-1} d_{f(\gamma)} M^{n-1}$  and  $M^n \cap d_{f(\gamma)}^{-1} \gamma M = M^n \cap d_{f(\gamma)}^{-1} \gamma M^{n+1}$ .*

*Consequently if  $C$  is any  $\{0, \dots, t\}$ -word, and  $X$  and  $Y$  are subspaces of  $e_{t(C)}M$  and  $e_{h(C)}M$  respectively, then we have  $Y^n \cap CX = Y^n \cap CX^{n+\mu_C(t)}$  for each  $n \in \mathbb{Z}$ .*

*Proof.* If  $f : X \rightarrow Y$  is a graded  $k$ -linear map of degree  $m \in \mathbb{Z}$  let  $(f)$  be the linear relation  $\{(x, f(x)) \mid x \in X\}$  from  $X$  to  $Y$  defined by the graph of  $f$ . Then we have  $Y \cap (f)X^{n-m} = Y^n \cap (f)X^{n-m} = Y^n \cap (f)X$  and  $X^{n-m} \cap (f)^{-1}Y = X^{n-m} \cap (f)^{-1}Y^n$  for any  $n \in \mathbb{Z}$  where  $X^i = X \cap M^i$  and  $Y^i = Y \cap M^i$  for each  $i \in \mathbb{Z}$ . This gives the first part of the corollary, and the second part follows by iteration.  $\square$

**Lemma 4.1.** *Let  $B$  and  $D$  be words with head  $v$  such that  $B^{-1}D$  is a word.*

- (i)  $B^+(M^\bullet) \cap D^+(M^\bullet) \cap e_v \text{rad}(M) \subseteq (B^+(M^\bullet) \cap D^-(M^\bullet)) + (B^-(M^\bullet) \cap D^+(M^\bullet))$ ,
- (ii)  $(B^-(M^\bullet) + D^+(M^\bullet) \cap B^+(M^\bullet)) \cap e_v \text{rad}(M) \subseteq (B^-(M^\bullet) + D^-(M^\bullet) \cap B^+(M^\bullet))$ ,
- (iii)  $B^+(M^\bullet) \cap D^\pm(M^\bullet) + e_v \text{rad}(M) = (B^+(M^\bullet) + e_v \text{rad}(M)) \cap (D^\pm(M^\bullet) + e_v \text{rad}(M))$ .

*Proof.* We just prove (i). (ii) and (iii) are similar, although lemma 3.4 is not required.

Recall  $\text{rad}(M) = \overline{A}M$ . For each  $\delta \in \{\pm 1\}$ , if it exists let  $x_\delta$  denote the arrow in  $Q$  with head  $v$  and sign  $\delta$ . If such an arrow doesn't exist let  $x_\delta = 0$ . For any  $m \in e_v \text{rad}(M)$  there are some  $m_1, m_{-1} \in M$  for which  $m = x_{-1}m_{-1} + x_1m_1$ .

By lemma 3.4 we have that  $x_1m_1 \in B^-(M^\bullet)$  and  $x_{-1}m_{-1} \in D^-(M^\bullet)$ . So if additionally  $m \in B^+(M^\bullet) \cap D^+(M^\bullet)$  we have  $x_1m_1 \in D^+(M^\bullet) \cap B^-(M^\bullet)$  as  $x_1m_1 = m - x_{-1}m_{-1}$  and  $D^-(M^\bullet) \subseteq D^+(M^\bullet)$ . By symmetry  $x_{-1}m_{-1} \in B^+(M^\bullet) \cap D^-(M^\bullet)$ .  $\square$

The next corollary follows from parts (i) and (ii) of lemma 4.1, and since  $\overline{\Lambda}/\text{rad}(\overline{\Lambda})$  has a basis in bijection with the vertices of  $Q$ , of which there are finitely many by assumption. Later we make use of part (iii) of lemma 4.1.

**Corollary 4.2.** *If  $M^\bullet$  is a complex of finitely generated projectives then the vector spaces  $F_{B,D,n}(M^\bullet)$  and  $G_{B,D,n}(M^\bullet)$  are finite-dimensional.*

Suppose now we restrict to the case where  $C = B^{-1}D$  is a periodic  $\mathbb{Z}$ -word of period  $p$ , say  $D = E^\infty$  and  $B = (E^{-1})^\infty$  for some  $\{0, \dots, p\}$ -word  $E$ . In this case we shall write  $C = {}^\infty E^\infty$ . In what follows we use results due to Crawley-Boevey [13, Section 4]) about linear relations between infinite-dimensional vector spaces.

**Definition 4.1.** ([13, Definitions 4.1 and 4.3]) If  $R$  is a linear relation on a vector space  $V$  (i.e. from  $V$  to itself) define the subspaces  $R^\sharp = R'' \cap (R^{-1})''$  and  $R^\flat = R'' \cap (R^{-1})' + R' \cap (R^{-1})''$  of  $V$  by setting  $R' = \bigcup_{n \in \mathbb{N}} R^n 0$  and  $R'' = \{v \in V : \exists v_0, v_1, v_2, \dots \in V \text{ with } v_0 = v \text{ and } v_i \in Rv_{i+1} \forall i\}$ .

*Proof of proposition 4.1.* (a): By lemmas 2.1 and 2.2 for any map of complexes  $f^\bullet : M^\bullet \rightarrow N^\bullet$  and any word  $C$  one has  $f(m) \in C^\pm(N^\bullet)$  for any  $m \in C^\pm(M^\bullet)$ . This together with the above corollary shows that for each  $i \in \mathbb{Z}$  the assignment  $\mathcal{C}_{\text{rad}}(\overline{\Lambda} - \mathbf{Proj}) \rightarrow k - \mathbf{Mod}$  sending  $M^\bullet$  to  $M^i \cap CM$  is functorial, where  $f^\bullet$  is sent to the restriction of  $f^i$  to  $M^i \cap CM$ , which maps into  $N^i \cap CN$ . As a result one may show refined functors define additive functors  $\mathcal{C}_{\text{rad}}(\overline{\Lambda} - \mathbf{Proj}) \rightarrow k - \mathbf{Mod}$ .

By parts (i) and (ii) of lemma 4.1 these functors send null-homotopic maps to the zero map, hence  $F_{B,D,n}$  and  $G_{B,D,n}$  define naturally isomorphic additive functors from  $\mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{Proj})$  to  $k - \mathbf{Mod}$ . The composition of including  $F_{B,D,n}^+(M^\bullet)$  into  $G_{B,D,n}^+(M^\bullet)$  and then projecting onto  $G_{B,D,n}(M^\bullet)$  sends any elements of  $F_{B,D,n}^-(M^\bullet)$  to 0, and the induced map defines a natural isomorphism  $F_{B,D,n} \rightarrow G_{B,D,n}$ .

(b): For  $n \in \mathbb{Z}$  we can consider the linear relation  $E(n) = E \cap (M^n \oplus M^n)$  on  $e_v M^n$ . Applying [13,

Lemma 4.5 ] gives an automorphism  $\theta^{E(n)}$  of  $E(n)^\sharp/E(n)^\flat$  by setting  $\theta^{E(n)}(m + E(n)^\flat) = m' + E(n)^\flat$  if and only if  $m' \in E(n)^\sharp \cap (E(n)^\flat + E(n)m)$ . By corollary 4.1 we have  $E(n)^\sharp = F_{B,D,n}^+(M^\bullet)$  and  $E(n)^\flat = F_{B,D,n}^-(M^\bullet)$  and so  $\theta^{E(n)}$  defines an action of  $T$  on  $F_{B,D,n}(M^\bullet)$  such that for an arbitrary map  $f^\bullet M^\bullet \rightarrow N^\bullet$  of complexes this action of  $T$  commutes with  $F_{B,D,n}(f^\bullet)$ .

(c) and (d): These are immediate consequences of the above and corollary 4.2.  $\square$

We now use some book-keeping in order to give a way of relating triples  $(B, D, n)$  and  $(B', D', n')$  so that the refined functors  $F_{B,D,n}$  and  $F_{B',D',n'}$  are naturally isomorphic. Later (lemmas 5.2 and 5.6) we shall see the converse: if  $F_{B,D,n}$  and  $F_{B',D',n'}$  are naturally isomorphic then  $(B, D, n)$  and  $(B', D', n')$  are related.

**Equivalence relations and the axis.** If  $C = B^{-1}D$  is a word we define the *axis*  $a_{B,D}$  of  $(B, D)$  as the unique integer satisfying  $C_{\leq a_{B,D}} = B^{-1}$  and  $C_{> a_{B,D}} = D$ . The following is tedious but obvious.

**Lemma 4.2.** *Suppose  $B$  and  $D$  are words such that  $C = B^{-1}D$  is a word.*

- (i) *If  $C$  is a  $\mathbb{Z}$ -word then  $a_{B,D} = 0$ .*
- (ii) *If  $C$  is a  $\mathbb{N}$ -word or a  $-\mathbb{N}$ -word then  $a_{D,B} = -a_{B,D}$ .*
- (iii) *If  $C$  is a  $\{0, \dots, t\}$ -word for  $t \in \mathbb{N}$  then  $a_{D,B} = t - a_{B,D}$ .*

Consider the set  $\Sigma$  consisting of all triples  $(B, D, n)$  where  $B^{-1}D$  is a word and  $n$  is an integer. Fix  $(B, D, n)$  and  $(B', D', n')$  from  $\Sigma$  and let  $C = B^{-1}D$  and  $C' = B'^{-1}D'$ . Recall that if  $C$  is not a  $\mathbb{Z}$ -word any shift of  $C$  is  $C$ . We say the words  $C$  and  $C'$  are *equivalent* when  $C'$  is either a shift  $C[m]$  of  $C$  or  $C'$  is a shift  $C^{-1}[m]$  of  $C^{-1}$  for some  $m \in \mathbb{Z}$ . In this case we define an integer

$$r(B, D; B', D') = \begin{cases} \mu_C(a_{B',D'}) - \mu_C(a_{B,D}) & \text{(if } C' = C \text{ is not a } \mathbb{Z}\text{-word)} \\ \mu_C(a_{D',B'}) - \mu_C(a_{B,D}) & \text{(if } C' = C^{-1} \text{ is not a } \mathbb{Z}\text{-word)} \\ \mu_C(\pm m) & \text{(if } C' = C^{\pm 1}[m] \text{ is a } \mathbb{Z}\text{-word)} \end{cases}$$

We introduce a relation  $\sim$  on  $\Sigma$  by setting  $(B, D, n) \sim (B', D', n')$  when  $B^{-1}D$  and  $B'^{-1}D'$  are equivalent and  $n' - n = r(B, D; B', D')$ . Now fix elements  $(B, D, n), (B', D', n'), (B'', D'', n'') \in \Sigma$  and let  $C = B^{-1}D$ ,  $C' = B'^{-1}D'$  and  $C'' = B''^{-1}D''$ . If  $C$  and  $C'$  are equivalent then  $r(B, D; B', D') = -r(B', D'; B, D)$ . Furthermore if  $C'$  and  $C''$  are all equivalent then

$$r(B, D; B'', D'') = r(B, D; B', D') + r(B', D'; B'', D'').$$

To see this use lemmas 1.1, 1.2 and 4.2. Consequently  $\sim$  on  $\Sigma$  defines an equivalence relation.

**The index set  $\mathcal{I}$ .** We let  $\Sigma(s)$  be the set of all  $(B, D, n) \in \Sigma$  where  $B^{-1}D$  is not a periodic  $\mathbb{Z}$ -word, and  $\Sigma(b)$  the set of  $(B, D, n) \in \Sigma$  where  $B^{-1}D$  is a periodic  $\mathbb{Z}$ -word. Note that for  $(B, D, n) \sim (B', D', n')$ ,  $(B, D, n) \in \Sigma(s)$  if and only if  $(B', D', n') \in \Sigma(s)$ . Consider the set  $\overline{\Sigma}$  of equivalence classes  $\overline{(B, D, n)}$  and the subsets  $\overline{\Sigma}(s)$  and  $\overline{\Sigma}(b)$  of  $\overline{\Sigma}$ .

Let  $\mathcal{I}(s)$  denote a chosen collection of representatives  $(B, D, n) \in \Sigma(s)$ , one for each class  $\overline{(B, D, n)} \in \overline{\Sigma}(s)$ . Similarly define the subset  $\mathcal{I}(b) \subseteq \Sigma(b)$  by choosing one representative  $(B, D, n) \in \Sigma(b)$  for each class  $\overline{(B, D, n)} \in \overline{\Sigma}(b)$ . Let  $\mathcal{I} = \mathcal{I}(s) \sqcup \mathcal{I}(b)$ . We now look at the symmetry in the definition of  $F_{B,D,n}$ .

The proof of the following lemma is similar to the proof for [13, Lemma 7.1].

**Lemma 4.3.** *Let  $B$  and  $D$  be words so that  $C = B^{-1}D$  is a word. Let  $n \in \mathbb{Z}$ .*

- (i) *If  $C$  is a non-periodic  $\mathbb{Z}$ -word then  $F_{B,D,n} \cong F_{D,B,n}$ .*
- (ii) *If  $C$  is a periodic  $\mathbb{Z}$ -word then  $F_{B,D,n} \cong \text{res}_t F_{D,B,n}$  as functors  $\mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{Proj}) \rightarrow k[T, T^{-1}] - \mathbf{Mod}$ .*

**Corollary 4.3.** *For and any  $n \in \mathbb{Z}$ , any words  $C, D = l_1^{-1}r_1 \dots l_s^{-1}r_s$ , and  $E$  such that  $C^{-1}DE$  is a word, the functors  $G_{C,DE,n}$  and  $G_{D^{-1}C,E,n+\mu_D(s)}$  are naturally isomorphic.*

*Proof.* With minor adjustments to the proof of [33, p.25, Lemma] one can show that for words  $C$  and  $E$  such that  $(d_{f(\gamma)}^{-1}\gamma C)^{-1}E$  is a word for  $\gamma \in \mathbf{Pa}_\rho$ ; the functors  $G_{C,E,n}$  and  $G_{C',E,n-1}$  are naturally isomorphic for any  $n \in \mathbb{Z}$  where  $C' = d_{f(\gamma)}^{-1}\gamma C$  and  $E' = \gamma^{-1}d_{f(\gamma)}E$ . Let  $C'' = \gamma^{-1}d_{f(\gamma)}C$

and  $E'' = d_{f(\gamma)}^{-1} \gamma E$ . By iteration it is enough to show  $G_{C,E'',n} \cong G_{C'',E,n+1}$  if  $C^{-1}(d_{f(\gamma)}^{-1} \gamma E)$  is a word. This follows by the above, lemma 4.3 (i), and proposition 4.1 (a).  $\square$

**Corollary 4.4.** *Suppose elements  $(B, D, n)$  and  $(B', D', n')$  from  $\mathcal{I}$  are equivalent.*

- (i) *If  $C$  is not a  $\mathbb{Z}$ -word then  $G_{B,D,n} \cong G_{B',D',n'}$ .*
- (ii) *If  $C$  is a  $\mathbb{Z}$ -word and  $C' = C[m]$  for some  $m \in \mathbb{Z}$  then  $G_{B,D,n} \cong G_{B',D',n'}$ .*
- (iii) *If  $C$  is a  $\mathbb{Z}$ -word and  $C' = C^{-1}[m]$  for some  $m \in \mathbb{Z}$  then  $G_{B,D,n} \cong \text{res}_i G_{B',D',n'}$  if  $C$  is periodic and  $G_{B,D,n} \cong G_{B',D',n'}$  otherwise.*

*Proof.* Follows by lemma 4.3 (ii) and corollary 4.3.  $\square$

For the purposes of the final proof it will be useful to give a functorial construction for complexes of the first and second kind.

**Constructive functors indexed by strings.** For  $(B, D, n) \in \mathcal{I}(s)$  we define a functor  $S_{B,D,n} : k - \mathbf{Mod} \rightarrow \mathcal{K}_{\text{rad}}(\bar{\Lambda} - \mathbf{Proj})$  as follows. On objects,  $S_{B,D,n}$  sends a vector space  $V$  to the complex  $S_{B,D,n}(V)$  whose homogeneous component is  $P^i(C)[\mu_C(a_{B,D}) - n] \otimes_k V$  and whose differential is  $d_{P(C)[\mu_C(a_{B,D}) - n]}^i \otimes 1_V$  in degree  $i \in \mathbb{Z}$ . That is  $S_{B,D,n}(V)$  may be considered the direct sum of  $P^\bullet(C)$  (shifted by  $\mu_C(a_{B,D}) - n$ ) indexed by a (possibly infinite) basis of  $V$ . For a linear map  $f : V \rightarrow V'$  and bases  $\{v_\lambda \mid \lambda \in \Omega\}$  for  $V$  and  $\{v'_{\lambda'} \mid \lambda' \in \Omega'\}$  for  $V'$ , write  $f(v_\lambda) = \sum a_{\lambda',\lambda} v'_{\lambda'}$  for scalars  $a_{\lambda',\lambda} \in k$  for each  $\lambda \in \Omega$ . Let  $b_{i,C}$  denote the coset of  $e_{v_C(i)}$  in the summand  $\bar{\Lambda} e_{v_C(i)}$  of  $P^\bullet(C)[\mu_C(a_{B,D}) - n]$ , and let  $b_{i,\lambda,C} = b_{i,C} \otimes v_\lambda$  for each  $i \in I_C$  and  $\lambda \in \Omega$ . Similarly for each  $\lambda' \in \Omega'$  let  $b'_{i,\lambda',C} = b_{i,C} \otimes v'_{\lambda'}$  for each  $i \in I_C$ . Define  $S_{B,D,n}$  on morphisms by extending the assignment  $S_{B,D,n}(f)(b_{i,\lambda,C}) = \sum_{\lambda' \in \Omega'} a_{\lambda',\lambda} b'_{i,\lambda',C}$  linearly over  $\bar{\Lambda}$ . Note that if  $B^{-1}D$  has controlled homogeneity then  $S_{B,D,n}$  defines a functor into  $\mathcal{K}_{\text{rad}}(\bar{\Lambda} - \mathbf{proj})$  upon restriction to  $k - \mathbf{mod}$ . The converse also holds.

**Constructive functors indexed by bands.** For  $(B, D, n) \in \mathcal{I}(b)$  we have  $a_{B,D} = 0$ . Furthermore the vector spaces  $V$  and  $V'$  have the additional structure of left  $k[T, T^{-1}]$ -modules and the  $k$ -linear map  $f : V \rightarrow V'$  above is additionally  $k[T, T^{-1}]$ -linear. Hence  $T$  defines automorphisms  $\varphi_V : V \rightarrow V$  and  $\varphi_{V'} : V' \rightarrow V'$  satisfying  $f\varphi_V = \varphi_{V'}f$ . Suppose  $C$  is periodic of period  $p$ . Here define the functor  $S_{B,D,n} : k[T, T^{-1}] - \mathbf{Mod} \rightarrow \mathcal{K}_{\text{rad}}(\bar{\Lambda} - \mathbf{Proj})$  on objects by  $S_{B,D,n}(V) = P^\bullet(C, V)[-n]$ . Note that the formula  $S_{B,D,n}(f)(b_{i,\lambda,C}) = \sum_{\lambda' \in \Omega'} a_{\lambda',\lambda} b'_{i,\lambda',C}$  defines a  $\bar{\Lambda} \otimes_k k[T, T^{-1}]$ -module morphism  $P^i(C, V)[-n] \rightarrow P^i(C, V')[-n]$  for each  $i \in \mathbb{Z}$  which defines a morphism of complexes. The proof of the following result will be written with the notation defined above.

**Corollary 4.5.** *Suppose elements  $(B, D, n)$  and  $(B', D', n')$  from  $\mathcal{I}$  are equivalent.*

- (i) *If  $C$  is not a periodic  $\mathbb{Z}$ -word then  $S_{B,D,n} \cong S_{B',D',n'}$ .*
- (ii) *If  $C$  is a periodic  $\mathbb{Z}$ -word and  $C' = C[m]$  for some  $m \in \mathbb{Z}$  then  $S_{B,D,n} \cong S_{B',D',n'}$ .*
- (iii) *If  $C$  is a periodic  $\mathbb{Z}$ -word and  $C' = C^{-1}[m]$  for some  $m \in \mathbb{Z}$  then  $S_{B,D,n} \cong S_{B',D',n'} \text{ res}_i$ .*

*Proof.* Suppose firstly  $C$  is not periodic. Here it is enough to define a bijection  $\omega : I_{C'} \rightarrow I_C$  such that the assignment  $b_{i,C'} \mapsto b_{\omega(i),C}$  (for each  $s \in \{1, \dots, d\}$ ,  $g \in \mathbb{Z}$ , and  $i \in \mu_{C'}^{-1}(g + \mu_{C'}(a_{B',D'}) - n')$ ) defines a morphism of complexes  $\theta^\bullet : P^\bullet(C')[\mu_{C'}(a_{B',D'}) - n'] \rightarrow P^\bullet(C)[\mu_C(a_{B,D}) - n]$ . Given the above and a vector space  $V$  with basis  $\{v_\lambda \mid \lambda \in \Omega\}$ , letting  $\sigma_V^r = \theta^r \otimes_k 1_V$  defines a  $\bar{\Lambda}$ -module map  $\sigma_V^r$  from  $P^r(C')[\mu_{C'}(a_{B',D'}) - n'] \otimes_k V$  to  $P^r(C)[\mu_C(a_{B,D}) - n] \otimes_k V$  which sends  $b_{i,\lambda,C'} = b_{i,C'} \otimes v_\lambda$  to  $b_{\omega(i),\lambda,C} = b_{\omega(i),C} \otimes v_\lambda$ . Furthermore this is a morphism of complexes since  $b_{\omega(i),C}^\pm \otimes v_\lambda = (\theta^r \otimes_k 1_V)(b_{i,C'}^\pm \otimes v_\lambda)$  for any  $i \in I_{C'}$ . By construction  $\sigma^\bullet$  defines a natural isomorphism. For the maps  $\omega$  one uses corollary 1.1.

Now suppose  $C$  is periodic. For (ii) and (iii) it is straightforward to check that the induced morphisms  $\sigma^\bullet$  commute with the action of  $T$  in the appropriate way.  $\square$

## 5. EVALUATION ON STRING AND BAND COMPLEXES.

An important part of the proof of completeness for the functorial filtration method is to apply our refined functors to the complexes of the form  $P^\bullet(C)$  or  $P^\bullet(C, V)$  defined in the introduction. Let  $\bar{C}^\pm(M^\bullet) = C^\pm(M^\bullet) + e_v \text{rad}(M)$  for any word  $C$ . By multiple applications of lemma 4.1 (iii) we have the following

$$\begin{aligned} F_{B,D,n}^+(M^\bullet) + e_v \text{rad}(M^n) &= M^n \cap (\bar{B}^+(M^\bullet) \cap \bar{D}^+(M^\bullet)), \\ F_{B,D,n}^-(M^\bullet) + e_v \text{rad}(M^n) &= M^n \cap ((\bar{B}^-(M^\bullet) \cap \bar{D}^-(M^\bullet)) + (\bar{B}^-(M^\bullet) \cap \bar{D}^+(M^\bullet))), \\ G_{B,D,n}^+(M^\bullet) + e_v \text{rad}(M^n) &= M^n \cap (\bar{B}^-(M^\bullet) + \bar{D}^+(M^\bullet) \cap \bar{B}^+(M^\bullet)), \\ G_{B,D,n}^-(M^\bullet) + e_v \text{rad}(M^n) &= M^n \cap (\bar{B}^-(M^\bullet) + \bar{D}^-(M^\bullet) \cap \bar{B}^+(M^\bullet)). \end{aligned}$$

By lemma 4.1 (i) the inclusion of  $F_{B,D,n}^\pm(M^\bullet)$  into  $F_{B,D,n}^+ + e_v \text{rad}(M^n)$  defines a natural isomorphism  $F_{B,D,n} \rightarrow \bar{F}_{B,D,n}$  and similarly by lemma 4.1 (ii) there is a natural isomorphism  $G_{B,D,n} \rightarrow \bar{G}_{B,D,n}$ . Consequently we have the following.

**Proposition 5.1.** *The functors  $F_{B,D,n}$ ,  $\bar{F}_{B,D,n}$ ,  $G_{B,D,n}$  and  $\bar{G}_{B,D,n}$  are all naturally isomorphic.*

So, to calculate  $F_{B',D',n'} S_{B,D,n}$  it will be enough to provide a description of  $A^\pm(M^\bullet) + \text{rad}(M)$  for any (finite or  $\mathbb{N}$ )-word  $A$ . To do this we adapt [13, Lemma 8.1] for our purposes, which will require some slightly involved calculations.

**String complexes.** For the next two lemmas we fix some notation. Recall that for each  $i \in I$  the symbol  $b_i$  denotes the coset of  $e_{v_C(i)}$  in the summand  $\bar{\Lambda} e_{v_C(i)}$  of  $P^{\mu_C(i)}(C)$ , and hence  $P(C)$  is generated as a  $\bar{\Lambda}$ -module by the elements  $b_i$ . For some arbitrary (but fixed)  $t \in I$  let  $\psi_t = \psi$  denote the  $\bar{\Lambda}$ -module epimorphism  $P(C) \rightarrow \bar{\Lambda} e_{v_C(t)}$  sending  $\sum_{i \in I} m_i$  to  $m_t$ .

**Notation.** For any  $j \in I$  let  $\mathbf{p}(j)$  denote the set of all paths  $\sigma_j \in \mathbf{Pa}_\rho$  with tail  $v_C(j)$ . For any  $\gamma \in \mathbf{Pa}_\rho$  write  $\mathbf{p}(\gamma, j)$  for the subset of  $\mathbf{p}(j)$  consisting of all  $\sigma_j$  with  $f(\sigma_j) = f(\gamma)$ . For each  $j \in I$  and each  $\sigma_j \in \mathbf{p}(j)$ , fix scalars  $\eta_j, \eta_{\sigma_j} \in k$  such that  $\sum_j \eta_j b_j + \sum_j \sum_{\sigma_j} \eta_{\sigma_j} \sigma_j b_j$  defines an element of  $P(C)$  where  $\sum_j \sum_{\sigma_j} \eta_{\sigma_j} \sigma_j b_j \in \text{rad}(P(C))$ . So the set of all  $j \in I$  where  $(\eta_j \neq 0 \text{ or } \eta_{\sigma_j} \neq 0 \text{ for some } \sigma_j \in \mathbf{p}(j))$  must be finite. However for each  $j \in I$  and  $\gamma \in \mathbf{Pa}_\rho$  the set  $\{\sigma_j \in \mathbf{p}(\gamma, j) \mid \eta_{\sigma_j} \neq 0\}$  may be infinite. Clearly we have  $\psi(\gamma \sum_{j \in I} \eta_j b_j) = \eta_t \gamma b_t$  and  $\psi(\gamma \sum_{j, \sigma_j} \eta_{\sigma_j} \sigma_j b_j) = \sum_{\sigma_t} \eta_{\sigma_t} \gamma \sigma_t b_t$ .

**Lemma 5.1.** *For any  $\gamma \in \mathbf{Pa}_\rho$  the map  $\psi$  sends  $d_{f(\gamma), P(C)}(\sum_{j \in I} \eta_j b_j)$  to*

$$\left\{ \begin{array}{ll} \eta_{t+1} \beta b_t & (\text{if } t+1 \in I, l_{t+1}^{-1} r_{t+1} = \beta^{-1} d_{f(\beta)} \text{ and } f(\beta) = f(\gamma)) \\ \eta_{t-1} \alpha b_t & (\text{if } t-1 \in I, l_t^{-1} r_t = d_{f(\alpha)}^{-1} \alpha \text{ and } f(\alpha) = f(\gamma)) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

and  $d_{f(\gamma), P(C)}(\sum_{j \in I} \sum_{\sigma_j \in \mathbf{p}(j)} \eta_{\sigma_j} \sigma_j b_j)$  to

$$\left\{ \begin{array}{ll} \sum_{\sigma_{t+1} \in \mathbf{p}(\gamma, t+1)} \eta_{\sigma_{t+1}} \sigma_{t+1} \beta b_t & (\text{if } t+1 \in I \text{ and } l_{t+1}^{-1} r_{t+1} = \beta^{-1} d_{f(\beta)}) \\ \sum_{\sigma_{t-1} \in \mathbf{p}(\gamma, t-1)} \eta_{\sigma_{t-1}} \sigma_{t-1} \alpha b_t & (\text{if } t-1 \in I \text{ and } l_t^{-1} r_t = d_{f(\alpha)}^{-1} \alpha) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

*Proof.* This is slightly involved, but essentially follows from the definition and lemma 2.2.  $\square$

If  $C = B^{-1}D$  is an  $I$ -word as above, for  $i \in I$  the words  $C_{>i}$  and  $(C_{\leq i})^{-1}$  have head  $v_C(i)$  and opposite sign. For  $\delta \in \{\pm 1\}$  let  $C(i, \delta)$  denote the one with sign  $\delta$ . If  $C(i, \delta) = C_{>i}$  then let  $d_i(C, \delta) = 1$ , and otherwise  $C(i, \delta) = (C_{\leq i})^{-1}$  in which case we let  $d_i(C, \delta) = -1$ . Consequently for any  $s \in I_{C(i, \delta)}$  we have  $C(i, \delta)_s = l_{i+s}^{-1} r_{i+s}$  if  $d_i(C, \delta) = 1$  and  $C(i, \delta)_s = r_{i-s+1}^{-1} l_{i-s+1}^{-1}$  if  $d_i(C, \delta) = -1$ . If  $C(i, \delta)$  is finite, there exists some  $n' \in \mathbb{N}$  for which  $C(i, \delta)$  is a  $\{0, \dots, n'\}$ -word. In which case choose  $u \in Q_0$  and  $\epsilon \in \{\pm 1\}$  so that  $C(i, \delta) 1_{u, \epsilon} = C(i, \delta)$ .

For scalars  $\mu, \mu' \in k$ ,  $\gamma \in \mathbf{Pa}_\rho$ , subsets  $\mathbf{q}$  and  $\mathbf{q}'$  of  $\mathbf{Pa}_\rho \setminus \{\gamma\}$  and scalars  $\mu_\omega, \mu'_{\omega'} \in k$  for each  $\omega \in \mathbf{q}$  and  $\omega' \in \mathbf{q}'$ , if  $\mu \gamma b_t + \sum_{\omega \in \mathbf{p}} \mu_\omega \omega b_t = \mu' \gamma b_t + \sum_{\omega' \in \mathbf{p}'} \mu'_{\omega'} \omega' b_t$  for some  $t \in I$  then as



$\gamma b_t \notin \text{span}_k\{\omega b_t, \omega' b_t \mid \omega \in \mathbf{p}, \omega' \in \mathbf{p}'\}$  we have  $\mu = \underline{\mu}'$ . This is because the set of all paths with tail  $v_C(t)$  defines a linearly independent subset of  $\overline{\Lambda}e_{v_C(t)}$ , however it does not define a basis. This argument will be used repeatedly in proving the following lemma. As above fix scalars  $\eta'_j, \eta'_{\sigma_j} \in k$  for each  $j \in I$  and each  $\sigma_j \in \mathbf{p}(j)$  such that  $\sum_j \eta'_j b_j + \sum_{j, \sigma_j} \eta'_{\sigma_j} \sigma_j b_j$  is another element of  $P(C)$  with  $\sum_{j, \sigma_j} \eta'_{\sigma_j} \sigma_j b_j \in \text{rad}(P(C))$ .

**Corollary 5.1.** *For  $A \in \mathcal{W}_{v, \delta}$  and any  $I$ -word  $C$  let  $I_{A,+} = \{i \in I \mid v_C(i) = v \text{ and } C(i, \delta) \leq A\}$  and  $I_{A,-} = \{i \in I \mid v_C(i) = v \text{ and } C(i, \delta) < A\}$ . Then*

$$A^\pm(P^\bullet(C)) + e_v \text{rad}(P(C)) = \text{span}_k\{b_i \mid i \in I_{A,\pm}\} + e_v \text{rad}(P(C)).$$

*Proof.* Let  $d = d_i(C, \delta)$ . Fix  $n \in \mathbb{Z}$  with  $i + d(n-1), i + dn \in I$ , so that  $n-1, n \in I_{C(i, \delta)}$ . Note firstly that for  $x = \sum_j \eta_j b_j + \sum_{j, \sigma_j} \eta_{\sigma_j} \sigma_j b_j$  and  $x' = \sum_j \eta'_j b_j + \sum_{j, \sigma_j} \eta'_{\sigma_j} \sigma_j b_j$  if  $x \in C(i, \delta)_n x'$  then  $\eta_{i+d(n-1)} = \eta'_{i+dn}$ . Furthermore if  $n'$  exists,  $x \in 1_{u, \epsilon}^-(P^\bullet(C))$  implies  $\eta_{i+dn'} = 0$ . To see this use lemma 5.1 and the remark above. Together these observations show that  $\sum_{j \in I} \eta_j b_j + m_0 \in C(i, \delta)^-(P^\bullet(C))$  implies  $\lambda_i = 0$ . By definition  $b_{i+d(n-1)} \in C(i, \delta)_n b_{i+dn}$ , and so  $b_i \in C(i, \delta)^+(P^\bullet(C))$  and if  $n'$  exists,  $b_{i+dn'} \in 1_{u, \epsilon}^+(P^\bullet(C))$  where  $u \in Q_0$  and  $\epsilon \in \{\pm 1\}$  are defined so that  $C(i, \delta)1_{u, \epsilon} = C(i, \delta)$ . This proof from here is essentially [13, Lemma 8.1]. In our set-up one uses the above, that  $A^\pm$  is functorial and proposition 3.1.  $\square$

For the next result we require an ordering on the functors  $G_{B,D,n}$ . To do so we recall the total order given on words by definition 3.2. For each vertex  $v$  order the pairs of words in  $\mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$  by setting  $(B, D) < (B', D')$  whenever  $B < B'$  or  $B = B'$  and  $D < D'$ . We can now evaluate our refined functors on string complexes.

**Lemma 5.2.** *Let  $n, n' \in \mathbb{Z}$  and for some words  $B$  and  $D$  let  $C = B^{-1}D$  be an  $I$ -word which is not a periodic  $\mathbb{Z}$ -word. Fix some words  $B'$  and  $D'$  such that  $C' = B'^{-1}D'$  is a word.*

(i) *If  $i \in I$  then  $a_{C(i,1), C(i,-1)} = i$  and*

$$\bar{F}_{C(i,1), C(i,-1), n}^+(P^\bullet(C)[\mu_C(i) - n]) = \bar{F}_{C(i,1), C(i,-1), n}^-(P^\bullet(C)[\mu_C(i) - n]) \oplus kb_i.$$

(ii) *If  $C' = C$  and  $n - n' = \mu_C(a_{B,D}) - \mu_C(a_{B',D'})$  then  $\bar{F}_{B',D',n'}(P^\bullet(C)[\mu_C(a_{B,D}) - n]) \cong k$ .*

(iii) *If  $(B, D, n)$  is not equivalent to  $(B', D', n')$  then  $\bar{F}_{B',D',n'}(P^\bullet(C)[\mu_C(a_{B,D}) - n]) = 0$ .*

*Proof.* This is essentially the same as [13, Lemma.8.2], however our notion of a periodic word is slightly different. Let  $I_{C(i, \delta), \Delta}$  be the set of  $j \in I$  with  $C(j, \delta) = C(i, \delta)$ . For (i) it is enough to assume  $l \in I_{C(i,1), \Delta} \cap I_{C(i,-1), \Delta}$  and  $\mu_C(l) = \mu_C(i)$ , and prove  $l = i$ . If  $C_{>i} = (C_{\leq l})^{-1}$  and  $(C_{\leq i})^{-1} = C_{>l}$  then  $C[l] = C^{-1}[i]$  which means  $C$  is a shift of its inverse, contradicting [13, Lemma 2.1]. Hence  $C_{>i} = C_{>l}$  and  $C_{\leq i} = C_{\leq l}$  which shows  $C = C[l - i]$ . Applying lemma 1.2 twice yields  $\mu_C(l - i) = 0$ . This shows that if  $l \neq i$  then  $C$  is periodic which is impossible.  $\square$

We now require a mapping property such as [33, p.27, Proposition]. For this we require the following book keeping which is essentially a corollary of lemma 1.1.

**Lemma 5.3.** *Let  $C$  be some  $I$ -word. Suppose  $C = B^{-1}D$  and  $i = a_{B,D}$ . Then for each  $j \in I$  we have*

$$\begin{aligned} (a) \quad v_C(j) &= \begin{cases} v_B(i - j) & \text{if } i - j \geq 0, \\ v_D(j - i) & \text{if } j - i \geq 0. \end{cases} & (b) \quad \mu_C(j) - \mu_C(i) &= \begin{cases} \mu_B(i - j) & \text{if } i - j \geq 0, \\ \mu_D(j - i) & \text{if } j - i \geq 0. \end{cases} \\ (c) \quad (((C_{\leq i})^{-1})_{i-j+1})^{-1} &= C_j \text{ for } i \geq j \text{ and } (C_{>i})_{j-i} = C_j \text{ for } i < j. \end{aligned}$$

**Lemma 5.4.** *Let  $(B, D, n) \in \mathcal{I}(s)$  and  $C = B^{-1}D$ . Let  $M^\bullet$  be a complex of projectives with radical images. Then for some basis  $\mathcal{B} = \{\bar{m}_\lambda \mid \lambda \in \Omega\}$  of  $F_{B,D,n}(M^\bullet)$  there is a morphism of complexes  $\theta_{B,D,n,M}^\bullet : \bigoplus_\lambda P^\bullet(C)[\mu_C(a_{B,D}) - n] \rightarrow M^\bullet$  such that  $F_{B,D,n}(\theta_{B,D,n,M}^\bullet)$  is an isomorphism.*

*Proof.* This proof follows the same idea as [13, Lemma 8.3]. For convenience we write  $i = a_{B,D}$ . For  $j \in I$  let  $b_{j,\lambda}$  denote the coset of  $e_{v_C(j)}$  in the summand  $P^{\mu_C(j)}(C)$  of  $\bigoplus_\lambda P(C)[\mu_C(i) - n]$  labeled by  $\lambda$ . To define a module map  $\bigoplus_B \bigoplus_{i \in I} \bar{\Lambda}_{e_{v_C(i)}} \rightarrow M$  it is enough to define elements  $m_{j,\lambda} \in e_{v_C(j)}M$  for each  $\lambda \in \Omega$  and each  $j \in I$ , and extend the assignment  $b_{j,\lambda} \mapsto m_{j,\lambda}$  linearly over  $\bar{\Lambda}$ . For each  $\lambda \in \Omega$  we can choose a lift  $m_{i,\lambda} \in F_{B,D,n}^+(M^\bullet) \setminus F_{B,D,n}^-(M^\bullet)$  of  $\bar{m}_\lambda$ . Since  $m_{i,\lambda} \in e_v M^n \cap B^+(M^\bullet)$  by corollary 4.1 there is  $m_{s,\lambda}^B \in e_{v_B(s)} M^{n+\mu_B(s)}$  for each  $s \in I_B$  where  $m_{s-1,\lambda}^B \in B_s m_{s,\lambda}^B$  given  $s-1 \in I_B$ . Similarly there exists  $m_{t,\lambda}^D \in e_{v_D(t)} M^{n+\mu_D(t)}$  for each  $t \in I_D$  where  $m_{t-1,\lambda}^D \in D_t m_{t,\lambda}^D$  given  $t-1 \in I_D$ . Set  $m_{j,\lambda} = m_{i-j,\lambda}^B$  whenever  $j \leq i$  and  $m_{j,\lambda} = m_{j-i,\lambda}^D$  whenever  $j \geq i$ . By lemma 5.3  $\theta_{B,D,n,M}^\bullet$  is a morphism of complexes and by lemma 5.2  $F_{B,D,n}(\theta_{B,D,n,M}^\bullet)$  is an isomorphism.  $\square$

To evaluate our refined functors on complexes of the form  $P^\bullet(C)$  we used that, for pairs  $(m, m')$  in a relation  $C(i, \delta)_n = (d_{f(\gamma)}^{-1} \gamma)^{\pm 1}$ , the coefficient of  $b_{i+d(n-1)}$  in  $m$  coincides with the coefficient of  $b_{i+dn}$  in  $m'$ . For the case  $P^\bullet(C, V)$  we will use a similar argument, however this case has extra complications.

**Band complexes.** Let  $V$  be a  $k[T, T^{-1}]$ -module with chosen basis  $\{v_\lambda \mid \lambda \in \Omega\}$ . Suppose also  $C$  is a periodic  $\mathbb{Z}$ -word of period  $p$ , say  $C = {}^\infty E^\infty$  for a word  $E = l_1^{-1} r_1 \dots l_p^{-1} r_p$ . Then for each  $i \in \{0, \dots, p-1\}$  and  $\lambda \in \Omega$  we use  $b_{i,\lambda}$  to denote the element  $b_i \otimes v_\lambda$  of  $P^{\mu_C(i)}(C, V)$ . Similarly to the case for string complexes, we now write  $\sigma_j$  for paths in  $\mathbf{Pa}_p$  with tail  $v_C(j)$ , and then choose scalars  $\eta_{j,\lambda}, \eta_{\sigma_j,\lambda} \in k$  for each  $\lambda \in \Omega$ ,  $j \in \{0, \dots, p-1\}$  and  $\sigma_j$  so that  $m = \sum_{j=0}^{p-1} \sum_{\lambda \in \Omega} (\eta_{j,\lambda} b_{j,\lambda} + \sum_{\sigma_j} \eta_{\sigma_j,\lambda} \sigma_j b_{j,\lambda})$  defines an element of  $P(C, V)$ . Recall there is an surjective module map  $\varphi_t : P(C, V) \rightarrow \bar{\Lambda}_{e_{v_C(t)}} \otimes V$  (for each  $t \in \{0, \dots, p-1\}$ ) defined by sending  $m$  to  $\sum_{\lambda \in \Omega} (\eta_{t,\lambda} b_{t,\lambda} + \sum_{\sigma_t} \eta_{\sigma_t,\lambda} \sigma_t b_{t,\lambda})$ .

**Corollary 5.2.** Fix  $t \in \{0, \dots, p-1\}$ . Then for any  $\xi \in Q_1$ ,  $m \in P(C)$  and  $v \in V$  we have  $d_{\xi, P(C,V)}(m \otimes v) = d_{\xi, P(C)}(m) \otimes v$ . Consequently for any  $\gamma \in \mathbf{Pa}_p$  the map  $\varphi_t$  sends  $\sum_{\lambda \in \Omega} d_{f(\gamma), P(C,V)}(\sum_{j=0}^{p-1} \eta_{j,\lambda} b_{j,\lambda})$  to

$$\left\{ \begin{array}{ll} \sum_\lambda \eta_{0,\lambda} \beta b_{p-1} \otimes T v_\lambda & (\text{if } t = p-1, l_p^{-1} r_p = \beta^{-1} d_{f(\beta)} \text{ and } f(\beta) = f(\gamma)) \\ \sum_\lambda \eta_{p-1,\lambda} \alpha b_0 \otimes T^{-1} v_\lambda & (\text{if } t = 0, l_0^{-1} r_0 = d_{f(\alpha)}^{-1} \alpha \text{ and } f(\alpha) = f(\gamma)) \\ \sum_\lambda \eta_{t+1,\lambda} \beta b_t \otimes v_\lambda & (\text{if } 0 \leq t < p-1, l_{t+1}^{-1} r_{t+1} = \beta^{-1} d_{f(\beta)} \text{ and } f(\beta) = f(\gamma)) \\ \sum_\lambda \eta_{t-1,\lambda} \alpha b_t \otimes v_\lambda & (\text{if } 0 < t \leq p-1, l_t^{-1} r_t = d_{f(\alpha)}^{-1} \alpha \text{ and } f(\alpha) = f(\gamma)) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

and  $\sum_{\lambda \in \Omega} d_{f(\gamma), P(C,V)}(\sum_{j=0}^{p-1} \sum_{\sigma_j} \eta_{\sigma_j,\lambda} \sigma_j b_{j,\lambda})$  to

$$\left\{ \begin{array}{ll} \sum_\lambda \sum_{\sigma_0 \in \mathbf{p}(\gamma,0)} \eta_{\sigma_0,\lambda} \sigma_0 \beta b_{p-1} \otimes T v_\lambda & (\text{if } t = p-1 \text{ and } l_p^{-1} r_p = \beta^{-1} d_{f(\beta)}) \\ \sum_\lambda \sum_{\sigma_{p-1} \in \mathbf{p}(\gamma,p-1)} \eta_{\sigma_{p-1},\lambda} \sigma_{p-1} \alpha b_0 \otimes T^{-1} v_\lambda & (\text{if } t = 0, \text{ and } l_0^{-1} r_0 = d_{f(\alpha)}^{-1} \alpha) \\ \sum_\lambda \sum_{\sigma_{t+1} \in \mathbf{p}(\gamma,t+1)} \eta_{\sigma_{t+1},\lambda} \sigma_{t+1} \beta b_t \otimes v_\lambda & (\text{if } 0 \leq t < p-1, \text{ and } l_{t+1}^{-1} r_{t+1} = \beta^{-1} d_{f(\beta)}) \\ \sum_\lambda \sum_{\sigma_{t-1} \in \mathbf{p}(\gamma,t-1)} \eta_{\sigma_{t-1},\lambda} \sigma_{t-1} \alpha b_t \otimes v_\lambda & (\text{if } 0 < t \leq p-1, \text{ and } l_t^{-1} r_t = d_{f(\alpha)}^{-1} \alpha) \\ 0 & (\text{otherwise}) \end{array} \right\}$$

*Proof.* The formula  $d_{\xi, P(C,V)}(m \otimes v) = d_{\xi, P(C)}(m) \otimes v$  is clear by definition. The second part of the corollary follows by lemma 5.1 and some very careful case analysis.  $\square$

Fix  $\gamma \in \mathbf{Pa}_p$  and for each  $\lambda \in \Omega$  fix arbitrary scalars  $\mu_\lambda, \mu'_\lambda \in k$ , subsets  $\mathbf{q}$  and  $\mathbf{q}'$  of  $\mathbf{Pa}_p \setminus \{\gamma\}$  and scalars  $\mu_{\omega,\lambda}, \mu'_{\omega',\lambda} \in k$  for each  $\omega \in \mathbf{q}$  and  $\omega' \in \mathbf{q}'$ . If

$$\sum_\lambda \mu_\lambda \gamma b_t \otimes T^r v_\lambda + \sum_\lambda \sum_{\omega \in \mathbf{p}} \mu_{\omega,\lambda} \omega b_t \otimes T^r v_\lambda = \sum_\lambda \mu'_\lambda \gamma b_t \otimes T^{r'} v_\lambda + \sum_\lambda \sum_{\omega' \in \mathbf{p}'} \mu'_{\omega',\lambda} \omega' b_t \otimes T^{r'} v_\lambda$$

for some  $t \in \{0, \dots, p-1\}$  and some  $r, r' \in \mathbb{Z}$ , then as  $\gamma b_t \otimes T^r v_\lambda$  does not lie in the span of the elements  $\omega b_t \otimes T^r v_\lambda$  and  $\omega' b_t \otimes T^{r'} v_\lambda$  with  $\omega \in \mathbf{q}$  and  $\omega' \in \mathbf{q}'$  we have  $\sum_\lambda \mu_\lambda \gamma b_t \otimes T^r v_\lambda = \sum_\lambda \mu'_\lambda \gamma b_t \otimes T^{r'} v_\lambda$ . In particular  $r = r'$  implies  $\sum_\lambda (\mu_\lambda - \mu'_\lambda) \gamma b_{t-pr, \lambda} \otimes T^r v_\lambda = 0$  in which case  $\mu_\lambda = \mu'_\lambda$  for each  $\lambda \in \Omega$ .

**Lemma 5.5.** Fix  $i \in \{0, \dots, p-1\}$ ,  $\mu \in \Omega$  and  $d = d_i(C, \delta)$ . Suppose for some  $n \in \{1, \dots, p\}$  we have

$$\left( \sum_{j, \lambda} \eta_{j, \lambda} b_{j, \lambda} + \sum_{j, \lambda, \sigma_j} \eta_{\sigma_j, \lambda} \sigma_j b_{j, \lambda} \right) \in C(i, \delta)_n \left( \sum_{j, \lambda} \eta'_{j, \lambda} b_{j, \lambda} + \sum_{j, \lambda, \sigma_j} \eta'_{\sigma_j, \lambda} \sigma_j b_{j, \lambda} \right) \quad (\dagger)$$

Then;

- (i) if  $(i+n < p \text{ and } d=1) \text{ or } (i-n+1 > 0 \text{ and } d=-1)$ , then  $\eta_{i+d(n-1), \mu} = \eta'_{i+dn, \mu}$ ,
- (ii) if  $(i+n > p \text{ and } d=1) \text{ or } (i-n+1 < 0 \text{ and } d=-1)$ , then  $\eta_{i+d(n-p-1), \mu} = \eta'_{i+d(n-p), \mu}$ ,
- (iii) when  $i+n = p$  and  $d=1$  we have  $\{\eta'_{0, \lambda} \mid \lambda \in \Omega\} = \{0\}$  iff  $\{\eta_{p-1, \lambda} \mid \lambda \in \Omega\} = \{0\}$ , and
- (iv) when  $i-n+1 = 0$  and  $d=-1$  we have  $\{\eta_{0, \lambda} \mid \lambda \in \Omega\} = \{0\}$  iff  $\{\eta'_{p-1, \lambda} \mid \lambda \in \Omega\} = \{0\}$ .

*Proof.* Note that in general we have  $i+n > 0$ ,  $i+n-p \leq p-1$ ,  $p-1 > i-n$ , and  $i-n+p \geq 0$ . We only prove (i) as (ii), (iii) and (iv) are similar.

If  $C(i, \delta)_n = d_{f(\gamma)}^{-1} \gamma$  then when  $d=1$  we have  $l_{i+n}^{-1} r_{i+n} = d_{f(\gamma)}^{-1} \gamma$  and  $0 < i+n \leq p-1$ , and when  $d=-1$  we have  $l_{i-n+1}^{-1} r_{i-n+1} = (C(i, \delta)_n)^{-1} = \gamma^{-1} d_{f(\gamma)}$  and  $0 < i-n+1 \leq p-1$ . In either case, applying corollary 5.2 with  $t = i+dn$  shows  $\eta_{i+d(n-1), \lambda} = \eta'_{i+dn, \lambda}$  for each  $\lambda \in \Omega$ , using the remark before the statement of the lemma.

Now suppose  $C(i, \delta)_n = \gamma^{-1} d_{f(\gamma)}$ . Similarly we have  $0 \leq i+n-1 < p-1$  and  $l_{i+n}^{-1} r_{i+n} = \gamma^{-1} d_{f(\gamma)}$  when  $d=1$ , and  $0 \leq i-n < p-1$  and  $l_{i-n+1}^{-1} r_{i-n+1} = d_{f(\gamma)}^{-1} \gamma$  when  $d=-1$ . So by applying corollary 5.2 with  $t = i+d(n-1)$  and  $\varphi_{i+d(n-1)}$  to  $(\dagger)$  shows  $\eta_{i+d(n-1), \lambda} = \eta'_{i+dn, \lambda}$  for all  $\lambda \in \Omega$ .  $\square$

**Corollary 5.3.** For  $A \in \mathcal{W}_{v, \delta}$  and any periodic  $\mathbb{Z}$ -word  $C$  of period  $p$  let  $I_{A, +}^p$  (resp.  $I_{A, -}^p$ ) be the set of all  $i \in \{0, \dots, p-1\}$  such that  $v_C(i) = v$  and  $C(i, \delta) \leq A$  (resp.  $C(i, \delta) < A$ ). Then

$$A^\pm(P^\bullet(C, V)) + e_v \text{rad}(P(C, V)) = \text{span}_k \{b_{i, \lambda} \mid \lambda \in \Omega, i \in I_{A, \pm}^p\} + e_v \text{rad}(P(C, V)).$$

*Proof.* Let  $i \in I$  be arbitrary. Note that  $b_{i, \lambda} \in C(i, \delta)^+(P^\bullet(C, V))$  by definition and corollary 5.2. By lemma 5.5 if  $\sum_{j, \lambda} \eta_{j, \lambda} b_{j, \lambda} + m \in C(i, \delta)^-(P^\bullet(C, V))$  with  $m \in \text{rad}(P(C, V))$  then  $\eta_{i, \lambda} = 0$  for each  $\lambda \in \Omega$ . The proof from here is similar to the proof of corollary 5.1.  $\square$

**Lemma 5.6.** Let  $n, n' \in \mathbb{Z}$  and for some cyclic word  $E = l_1^{-1} r_1 \dots l_p^{-1} r_p$  let  $B = (E^{-1})^\infty$  and  $D = E^\infty$  such that  $C = B^{-1} D$  is a periodic  $\mathbb{Z}$ -word of period  $p$ . Fix some words  $B'$  and  $D'$  such that  $C' = B'^{-1} D'$  is a word. Let  $b_i \otimes V = V_i$  for each  $i \in \{0, \dots, p-1\}$ .

- (i) For any  $i \in \{0, \dots, p-1\}$  we have  $a_{C(i, 1), C(i, -1)} = i$  and

$$\bar{F}_{C(i, 1), C(i, -1), n}^+(P^\bullet(C, V)[\mu_C(i) - n]) = \bar{F}_{C(i, 1), C(i, -1), n}^-(P^\bullet(C, V)[\mu_C(i) - n]) \oplus V_i.$$

- (ii) If  $C' = C[m]$  and  $n - n' = \mu_C(m)$  for some  $m \in \mathbb{Z}$  then there's a  $k[T, T^{-1}]$ -module isomorphism  $\bar{F}_{B', D', n'}(P^\bullet(C, V)[-n]) \cong V$ .

- (iii) If  $(B, D, n)$  is not equivalent to  $(B', D', n')$  then  $\bar{F}_{B', D', n'}(P^\bullet(C, V)[-n]) = 0$ .

*Proof.* Similar to lemma 5.2.  $\square$

**Split linear relations.** We recall terminology from [13, p. 9]. For a relation  $R$  on a vector space  $V$  we say  $R$  is *split* if  $R^b$  has a complement  $U$  where for each  $u \in U$  the set  $U \cap Ru$  has exactly one element,  $\varphi_R(u)$ . For a complex  $M^\bullet$  and a relation  $R$  on the vector space  $e_v M$  define the relation  $R(n) = R \cap (M^n \oplus M^n)$  on  $e_v M^n$ . We say  $M^\bullet$  is *R-split* if  $R(n)$  is split for each  $n \in \mathbb{Z}$ . Note  $F_{B, D, n}(M^\bullet)$  is finite dimensional by corollary 4.2. By [13, Lemma 4.6] this is sufficient to prove the following corollary, as  $E(n)^\sharp / E(n)^b = F_{B, D, n}(M^\bullet)$ .

**Corollary 5.4.** *Let  $M^\bullet$  be a complex of finitely generated projectives with radical images. Then if  $C$  is a periodic  $\mathbb{Z}$ -word of period  $p > 0$ , say  $C, M^\bullet$  is  $E$ -split.*

**Lemma 5.7.** *Let  $(B, D, n) \in \mathcal{I}(b)$  and so  $C = B^{-1}D$  is a periodic  $\mathbb{Z}$ -word of period  $p > 0$ , say  $C = {}^\infty E^\infty$ . Let  $M^\bullet$  be a complex of projectives with radical images for which the relation  $E(n)$  on  $M^n$  is split. Let  $V = F_{B,D,n}(M^\bullet)$ . Then there is a morphism of complexes  $\theta_{B,D,n,M}^\bullet : P^\bullet(C, V)[-n] \rightarrow M^\bullet$  in  $\mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{Proj})$  such that  $F_{B,D,n}(\theta_{B,D,n,M}^\bullet)$  is an isomorphism.*

*Proof.* Using corollary 4.1 there are elements  $u_{0,\lambda}, \dots, u_{p,\lambda} \in M$  where;  $u_{j,\lambda} \in e_{v_E(j)} M^{n+\mu_E(j)}$  for each  $j \in \{0, \dots, p\}$ ,  $u_{p,\lambda} = u_{0,\lambda}$ ,  $u_{0,\lambda} = \varphi_{E(n)}(u_\lambda)$ , and  $u_{j-1,\lambda} \in l_j^{-1} r_j u_{j,\lambda}$  given  $j > 0$ . Recall that  $P(C, V)$  is freely generated by the elements  $b_j \otimes \bar{u}_\lambda$  for  $\lambda \in \Omega$  and  $j \in \{0, \dots, p-1\}$ . So to define a  $\overline{\Lambda}$ -module map  $\theta_{B,D,n,M} : P(C, V)[-n] \rightarrow M$  it is enough to let  $\theta_{B,D,n,M}(b_j \otimes \bar{u}_\lambda) = u_{j,\lambda}$  for each  $\lambda \in \Omega$  and  $j \in \{0, \dots, p-1\}$ . Using lemma 2.2 one can show  $\theta_{B,D,n,M}^\bullet : P^\bullet(C, V)[-n] \rightarrow M^\bullet$  defines a map of complexes. By lemma 5.6  $F_{B,D,n}(\theta_{B,D,n,M}^\bullet)$  is an isomorphism.  $\square$

## 6. DIRECT SUMS OF STRING AND BAND COMPLEXES.

Throughout this section we fix a complex

$$N^\bullet = \left( \bigoplus_{\sigma \in \mathcal{S}} P^\bullet(A^\sigma)[-t_\sigma] \right) \oplus \left( \bigoplus_{\beta \in \mathcal{B}} P^\bullet(E^\beta, V^\beta)[-s_\beta] \right)$$

for index sets  $\mathcal{S}$  and  $\mathcal{B}$  where  $t_\sigma \in \mathbb{Z}$  and  $A^\sigma$  is a non-periodic word for each  $\sigma$ , and  $s_\beta \in \mathbb{Z}$  and  $E^\beta$  is a periodic  $\mathbb{Z}$  word for each  $\beta$ . For  $n \in \mathbb{Z}$  and words  $B$  and  $D$  such that  $C = B^{-1}D$  is a word. Let  $\mathcal{S}(B, D, n)$  be the set of all  $\sigma \in \mathcal{S}$  where  $(B, D, n)$  and  $((A_{\leq 0}^\sigma)^{-1}, A_{> 0}^\sigma, t_\sigma)$  are equivalent. Let  $\mathcal{B}(B, D, n)^\pm$  be the set of all  $\beta \in \mathcal{B}$  such that  $E^\beta = C^{\pm 1}[m]$  and  $s_\beta - n = \mu_C(\pm m)$  for some  $m \in \mathbb{Z}$ .

Let  $\overline{\mathcal{W}}$  denote the set of all equivalence classes of words. For each equivalence class  $\overline{C}$  of  $\overline{\mathcal{W}}$  we choose one representative  $C$  and one pair of words  $(B, D)$  for which  $B^{-1}D = C$ . Let  $\mathcal{W}$  be the set of these chosen pairs  $(B, D)$ . Let  $\mathcal{W}(s)$  be the subset of  $\mathcal{W}$  consisting of all pairs  $(B, D)$  for which  $B^{-1}D$  is not a periodic  $\mathbb{Z}$ -word, and let  $\mathcal{W}(b) = \mathcal{W} \setminus \mathcal{W}(s)$ . For  $\beta \in \mathcal{B}(B, D, n)^-$  let  $U^\beta = \text{res}_\ell V^\beta$ .

**Theorem 6.1.** *Let  $N^\bullet$  be an arbitrary direct sum of string and band complexes as above, and let  $n' \in \mathbb{Z}$  and  $(B', D') \in \mathcal{W}$  be arbitrary.*

(I) *If  $(B', D') \in \mathcal{W}(s)$  there are  $\dim(F_{B',D',n'}(N^\bullet))$  elements in  $\sigma \in \mathcal{S}(B', D', n')$ .*

(II) *If  $(B', D') \in \mathcal{W}(b)$  then  $F_{B',D',n'}(N^\bullet) \cong \bigoplus_{\beta_+} V^{\beta_+} \oplus \bigoplus_{\beta_-} U^{\beta_-}$  where  $\beta_\pm$  runs through  $\mathcal{B}(B', D', n')^\pm$ .*

*Consequently given  $(B, D) \in \mathcal{W}$  is fixed and  $C = B^{-1}D$ ,*

(a) *if  $(B, D) \in \mathcal{W}(s)$  there are  $\sum_{n' \in \mathbb{Z}} \dim(F_{B,D,n'}(N^\bullet))$  elements  $\sigma \in \mathcal{S}$  where  $A^\sigma = C$  or  $A = C^{-1}$ ,*

(b) *if  $(B, D) \in \mathcal{W}(b)$  then  $\bigoplus_{n' \in \mathbb{Z}} F_{B,D,n'}(N^\bullet) \cong \bigoplus_{\gamma_+} V^{\gamma_+} \oplus \bigoplus_{\gamma_-} U^{\gamma_-}$  where  $\gamma_\pm$  runs through  $\beta \in \mathcal{B}$  where  $E^\beta$  is a shift of  $C^{\pm 1}$ ;*

*and if instead  $n \in \mathbb{Z}$  is fixed then;*

(c) *there are  $\sum_{(B',D') \in \mathcal{W}(s)} \dim(F_{B',D',n}(N^\bullet))$  elements  $\sigma \in \mathcal{S}$  where  $P^{n-t_\sigma}(A^\sigma)[-t_\sigma] \neq 0$ ,*

(d)  *$\bigoplus_{(B',D') \in \mathcal{W}(b)} F_{B',D',n}(N^\bullet) \cong \bigoplus_{\alpha_+} V^{\alpha_+} \oplus \bigoplus_{\alpha_-} U^{\alpha_-}$  where  $\alpha_+$  (resp.  $\alpha_-$ ) runs through  $\beta \in \mathcal{B}$  with  $P^{n-s_\beta}(E^\beta, V^\beta) \neq 0$  (resp.  $P^{n-s_\beta}(E^\beta, U^\beta) \neq 0$ ).*

Proving this theorem will require the following.

**Corollary 6.1.** *Fix words  $B$  and  $D$  for which  $C = B^{-1}D$  is a word. Then the union  $\bigcup_{n \in \mathbb{Z}} \mathcal{S}(B, D, n)$  (resp.  $\bigcup_{n \in \mathbb{Z}} \mathcal{B}(B, D, n)^\pm$ ) is disjoint and consists of all  $\sigma \in \mathcal{S}$  (resp.  $\beta \in \mathcal{B}$ ) for which  $A^\sigma$  is equivalent to (resp.  $(E^\beta)^{\pm 1}$  is a shift of)  $C$ .*

*Proof.* Straightforward.  $\square$

**Corollary 6.2.** *Fix  $n \in \mathbb{Z}$ . Then the union  $\bigcup_{(B,D) \in \mathcal{W}(s)} \mathcal{S}(B, D, n)$  (resp.  $\bigcup_{(B,D) \in \mathcal{W}(b)} \mathcal{B}(B, D, n)^+$  or  $\bigcup_{(B,D) \in \mathcal{W}(b)} \mathcal{B}(B, D, n)^-$ ) is disjoint and consists of all  $\sigma \in \mathcal{S}$  (resp.  $\beta \in \mathcal{B}$ ) for which  $P^n(A^\sigma)[-t_\sigma] \neq 0$  (resp.  $P^n(E^\beta, V^\beta)[-s_\beta] \neq 0$  or  $P^n(E^\beta, U^\beta)[-s_\beta] \neq 0$ ).*

*Proof.* Follows by definition and corollary 4.5.  $\square$

The following argument is essentially [13, Theorem 9.1].

*Proof of theorem 6.1.* For (I) use lemma 5.2 and corollary 4.4. For (II) use the same corollary and lemma 5.6. For part (a) (resp. (b)) use part (I) (resp. (II)) and corollary 6.1. For part (c) (resp. (d)) use (I) (resp. (II)) and corollary 6.2.  $\square$

**Theorem 6.2.** *Let  $M^\bullet$  be a complex of finitely generated projectives with radical images. Then there is a morphism of complexes  $\theta^\bullet : L^\bullet \rightarrow M^\bullet$  where  $L^\bullet$  is a direct sum of string and band complexes such that  $F_{B,D,n}(\theta^\bullet)$  is an isomorphism for each  $(B, D, n)$  from the index set  $\mathcal{I}$ .*

*Proof.* This is essentially the same as the proof for [13, Theorem 9.2]. Here we use lemmas 5.4 and 5.7.  $\square$

**Lemma 6.1.** *Let  $\theta^\bullet : N^\bullet \rightarrow M^\bullet$  be a chain map between complexes of projectives  $\overline{\Lambda}$ -modules with radical images, where  $N^\bullet$  is a direct sum of string and band complexes as above. Suppose  $F_{B,D,n}(\theta^\bullet)$  is injective for each  $n \in \mathbb{Z}$  and each pair of words  $(B, D)$  for which  $B^{-1}D$  is a word. Then  $\theta^i$  is injective for each  $i \in \mathbb{Z}$ .*

*Proof.* Similar to [13, Lemma 9.4].  $\square$

## 7. LINEAR COMPACTNESS AND THE COVERING PROPERTY.

To give a functorial filtration of  $\mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{proj})$  we now show how our functors *cover* each complex. This was done by Crawley-Boevey [13, p. 11, Lemma 5.2] and we follow this idea closely. With this in mind, we consider a slightly different topology on  $\overline{\Lambda}$  which is equivalent to the  $A$ -adic topology, and easier to work with.

Recall that  $\Lambda$  denotes the  $k$ -algebra  $kQ/(\rho)$ , where  $\overline{\Lambda} = \overline{kQ}/(\overline{\rho})$  is a gentle algebra.

**Primitive Cycles.** By a *primitive cycle* (as in [13, p.9]) we mean a cycle  $z_v$  (with head and tail  $v \in Q_0$ ) which is not a product of shorter cycles and where  $z_v^n \notin (\rho)$  for each  $n \in \mathbb{N}$ . In what follows let  $z = \sum_v z_v$  so that  $k[z]$  defines a sub-ring of the centre of  $\Lambda$  [13, p.4, Lemma 4.1]. So  $k[[z]]$  defines a sub-ring of the centre of  $\overline{\Lambda}$ . Recall  $A$  is the ideal of  $\Lambda$  generated by the arrows.

**Lemma 7.1.** *There exists some  $t \in \mathbb{N}_+$  for which  $A^t \subseteq (z)$ . Consequently:*

- (i)  $\overline{\Lambda}$  is isomorphic to the completion of  $\Lambda$  with respect to the  $(z)$ -adic topology.
- (ii) Any finitely generated projective  $\overline{\Lambda}$ -module  $X$  is a finitely generated  $k[[z]]$ -module.
- (iii)  $\overline{\Lambda}$  is a left Noetherian ring.

*Proof.* For each path  $\omega$  write  $l(\omega)$  for its length. Consider the set  $\Phi'$  of all paths  $\omega \in \mathbf{Pa}_\rho$  which lie outside the ideal  $(\Phi)$  generated by the set  $\Phi$  of all primitive cycles. Note  $\Phi$  is finite since  $Q_0$  is finite and  $\overline{\Lambda}$  is gentle.

$\Lambda$  is more generally (as in [13, Definition 1.1]) a string algebra, so by [13, p. 9, Lemma 4.1]  $\Lambda$  is a finitely generated  $k[z]$  module (since  $Q_0$  is finite) which shows  $\Phi'$  must be finite. It is straightforward to show  $A^t \subseteq (z)$  where  $t = \max\{l(\omega) \mid \omega \in \Sigma \sqcup \Sigma'\} + 1$ .

By the above the  $A$  adic and  $(z)$ -adic topologies on  $\Lambda$  are equivalent which gives (i). For (ii) we can assume  $X = \overline{\Lambda}$  and then use (i), [13, p. 9, Lemma 4.1], [3, p. 108, Propositions 10.12 and 10.13]. (iii) is immediate by (i) and (ii).  $\square$

**Convention.** Between now and the statement of lemma 7.3; modules are  $k[[z]]$ -modules, module maps are  $k[[z]]$ -module homomorphisms and the assumed topology is the  $(z)$ -adic topology.

**Linear Compactness.** We now recall and use the notion of linear compactness following Zelinsky [37, p. 80]. A *linear variety* (as in [37]) of a module  $N$  is a closed coset  $U + m$  of any sub-module  $U$  and we say  $N$  is *linearly compact* when any collection of closed linear varieties in  $N$  with the finite intersection property must have a non-void intersection. We start by highlighting some topological properties of complexes  $M^\bullet$  of finitely generated projectives with radical images. By [13, Lemma 4.1]  $z$  commutes with the action of  $\Lambda$ .

**Lemma 7.2.** *Fix  $v \in Q_0$ ,  $i \in \mathbb{Z}$ ,  $\alpha \in Q_1$ ,  $\gamma \in \mathbf{Pa}_\rho$ , and  $m \in e_v M^i$ .*

- (i) *The module  $e_v M^i$  is linearly compact.* (ii) *The map  $d_{\alpha, M} : e_v M \rightarrow e_v M$  is  $k[[z]]$ -linear.*
- (iii) *For  $n > 0$  and a sub-module  $U$  of  $e_v M^i$  with  $e_v(z)^n M^i \subseteq U$ ,  $U + m$  is closed.*
- (iv) *The coset  $\{m\} = 0 + m$  is closed.*
- (v) *If  $f : X \rightarrow Y$  is a module map and  $X$  and  $Y$  are linearly compact, then for closed sub-modules  $V$  of  $X$  and  $U$  of  $Y$  the sub-modules  $f^{-1}(U)$  of  $X$  and  $f(V)$  of  $Y$  are closed.*

*Proof.* For (i) use lemma 7.1 and [37, Propositions 1, 2, 4 and 5]. (ii) is a straightforward application of lemma 2.2. For (iii) and (iv) it is enough (say by [27, p. 98, Corollary 17.7]) to choose a limit point  $l$  of  $U + m$  and show  $l \in U + m$ . This follows by the assumption for (iii) and is clear for (iv).. For (v) we use [37, Propositions 2, 3 and 7].  $\square$

**Corollary 7.1.** *Fix; a vertex  $v$ ,  $\delta \in \{\pm 1\}$ ,  $i \in \mathbb{Z}$ , a subset  $U$  of  $M$ , and an  $I$ -word  $C = l_1^{-1} r_1 \dots$  from  $\mathcal{W}_{v, \delta}$ .*

- (i) *If  $I = \{0, \dots, t\}$  and  $M^{i+\mu_C(t)} \cap M'$  is a closed submodule of  $M^{i+\mu_C(t)}$  then  $e_v M^i \cap C M'$  is a closed submodule of  $e_v M^i$ .*
- (ii) *If  $I = \mathbb{N}$  and  $0 < t \in \mathbb{N}$  is fixed, for each  $m_{t-1} \in M^{i+\mu_C(t-1)} \cap \bigcap_{n>t-1} (C_{>t-1})_{\leq n} M$  there exists  $m_t \in M^{i+\mu_C(t)} \cap \bigcap_{n>t} (C_{>t})_{\leq n} M$  where  $m_{t-1} \in l_t^{-1} r_t m_t$ .*
- (iii) *If  $I = \mathbb{N}$  and  $U \subseteq e_v M^i$  we have  $U \cap C^+(M^\bullet) = U \cap \bigcap_{n \in \mathbb{N}} C_{\leq n} M$ .*
- (iv)  *$e_v M^i \cap C^-(M^\bullet)$  and  $e_v M^i \cap C^+(M^\bullet)$  are closed submodules of  $e_v M^i$ .*

*Proof.* Note firstly that left multiplication  $\gamma \times : e_u M^i \rightarrow e_v M^i$  by a path  $\gamma \in \mathbf{Pa}_\rho$  (with head  $v$  and tail  $u$ ) defines a module map by [13, p.6, Lemma 3.1]. By lemma 7.2 one can then show that the sub-modules  $\gamma U$ ,  $d_\alpha V$ ,  $d_\alpha^{-1} V \cap e_{h(\alpha)} M^{i-1}$  and  $\gamma^{-1} W \cap e_{t(\gamma)} M^i$  are closed sub-modules of  $e_{h(\gamma)} M^i$ ,  $e_{h(\alpha)} M^{i+1}$ ,  $e_{h(\alpha)} M^{i-1}$  and  $e_{t(\gamma)} M^i$  respectively. Consequently, for  $m \in e_{t(\gamma)} M^i$  and  $m' \in e_{h(\gamma)} M^i$  if the sets  $X = \gamma^{-1} d_{f(\gamma)} m \cap e_{h(\gamma)} M^{i+1}$  and  $Y = d_{f(\gamma)}^{-1} \gamma m' \cap e_{t(\gamma)} M^{i-1}$  are non-empty then they are closed cosets as  $\gamma m' = \{\gamma m'\}$  and  $d_{f(\gamma)} m = \{d_{f(\gamma), M}(m)\}$  are closed by lemma 7.2. We repeatedly apply this remark in what follows. (i) follows by corollary 4.1 and the above.

For each  $n \geq t$  we have  $m_{t-1} \in (C_{>t-1})_{\leq n} M$  and so there is some  $u_n \in M^{i+\mu_C(t)} \cap r_t^{-1} l_t m_{t-1}$  for which  $u_n \in (C_{>t})_{\leq n} M$ . Consider the collection  $\Delta$  consisting of all  $M^{i+\mu_C(t)} \cap (C_{>t})_{\leq n} M$  where  $n \geq t$ , together with the set  $M^{i+\mu_C(t)} \cap r_t^{-1} l_t m_{t-1}$ . By the above  $\Delta$  is a collection of closed cosets of  $e_{v_C(t)} M^{i+\mu_C(t)}$ , and from here we use that  $e_{v_C(i)} M^{j+\mu_C(i)}$  is linearly compact (see lemma 7.2). This gives (ii). (iii) is by definition, and (iv) will hold by definition, the above and lemma 7.2.  $\square$

**Lemma 7.3.** *Let  $\theta^\bullet : N^\bullet \rightarrow M^\bullet$  be a chain map between complexes of projectives  $\overline{\Lambda}$ -modules with radical images, and suppose each homogeneous component  $M^i$  of  $M^\bullet$  is finitely generated. Suppose  $F_{B,D,n}(\theta^\bullet)$  is surjective for each  $n \in \mathbb{Z}$  and each pair of words  $(B, D)$  for which  $B^{-1}D$  is a word. Then  $\theta^i$  is surjective for each  $i \in \mathbb{Z}$ .*

*Proof.* For the moment fix a vertex  $v$ , an integer  $r$  and some  $\delta \in \{\pm 1\}$ . Adapting the argument used to prove [13, Lemma 10.3], one may show that for any non-empty subset  $S$  of  $e_v M^r$  which does not meet  $\text{rad}(M)$  there is a word  $C \in \mathcal{W}_{v, \delta}$  such that either:  $C$  is finite and  $S$  meets  $C^+(M^\bullet)$  but not  $C^-(M^\bullet)$ , or  $C$  is an  $\mathbb{N}$ -word and  $S$  meets  $C_{\leq n} M$  but not  $C_{\leq n} \text{rad}(M)$  for each  $n \in \mathbb{N}$ . Extra complications in our setting are dealt with using corollaries 2.1 and 2.2 and lemmas 2.2 and 3.2.

Now suppose  $U$  is a  $k[[z]]$ -submodule of  $e_v M^r$  for which  $e_v \text{rad}(M^r) \subseteq U$ . Then if  $H$  is a subset of  $e_v M^r$  and  $m$  is an element from  $H$  which is not in  $U$  there is a word  $C \in \mathcal{W}_{v,\delta}$  such that  $H \cap (U + m)$  meets  $C^+(M^\bullet)$  but not  $C^-(M^\bullet)$ . To see this one may adapt the argument from the proof of [13, Lemma 10.4] without too many complications. This involves applying the above and corollary 7.1. One may similarly adapt [13, Lemma 10.5] to show that if  $m$  is an element in  $e_v M^r$  but not in  $U$ , there are words  $B \in \mathcal{W}_{v,\delta}$  and  $D \in \mathcal{W}_{v,-\delta}$  such that  $U + m$  meets  $G_{B,D,r}^+(M^\bullet)$  but not  $G_{B,D,r}^-(M^\bullet)$ .

Now for a contradiction suppose that  $\theta^i$  is not surjective for some  $i \in \mathbb{Z}$ . Since  $\text{rad}(M^i)$  is a superfluous sub-module of  $M^i$  we have  $e_v \text{im}(\theta^i) + e_v \text{rad}(M^i) \neq e_v M^i$  for some vertex  $v$ . Hence  $e_v \text{im}(\theta^i) + e_v \text{rad}(M^i)$  is contained in a maximal  $k[[z]]$ -submodule  $U$  of  $e_v M^i$ . From here the proof is similar to the proof of [13, Lemma 10.6]  $\square$

## 8. PROOFS OF THE MAIN RESULTS.

We start with a lemma concerning our functorial set-up considered by Gabriel [19] and Ringel [33]. In what follows all categories and functors are additive. A functor  $G$  between categories with arbitrary direct sums is said to *essentially commute with direct sums* if for each collection  $\{X_j\}_{j \in \mathcal{J}}$  of objects in the domain there are isomorphisms  $\sigma_X : G(\bigoplus_{j \in \mathcal{J}} X_j) \rightarrow \bigoplus_{j \in \mathcal{J}} G(X_j)$  satisfying  $\sigma_Y G(\bigoplus_{j \in \mathcal{J}} f_j) = (\bigoplus_{j \in \mathcal{J}} G(f_j)) \sigma_X$  for each collection of morphisms  $\{f_j\}_{j \in \mathcal{J}}$ .

**Lemma 8.1.** *Let  $\mathfrak{N}$  be a full subcategory of a category  $\mathfrak{M}$  which has arbitrary direct sums. For each  $i \in I$  let  $\mathfrak{A}_i$  be a full subcategory of a category  $\mathfrak{B}_i$  which has arbitrary direct sums, and let  $S_i : \mathfrak{B}_i \rightarrow \mathfrak{M}$  and  $F_i : \mathfrak{M} \rightarrow \mathfrak{B}_i$  be functors such that the restriction  $F_i|$  of  $F_i$  to  $\mathfrak{N}$  defines a functor with domain  $\mathfrak{A}_i$ . Furthermore, suppose that the following conditions hold;*

- (i) *for each  $i \in I$  we have  $F_i S_i \simeq 1_{\mathfrak{A}_i}$  and  $F_j S_i \simeq 0$  for each  $j \in I$  with  $j \neq i$ ,*
- (ii)  *$F_i$  essentially commutes with direct sums for each  $i \in I$ ,*
- (iii) *For every object  $M$  in  $\mathfrak{N}$  and each  $i \in I$  there is a map  $\gamma_{i,M} : S_i(F_i|)(M) \rightarrow M$  such that  $F_i(\gamma_{i,M})$  is an isomorphism;*
- and given any object  $M$  in  $\mathfrak{M}$  and any morphism  $\theta : \bigoplus_{i \in I} S_i(M_i) \rightarrow M$  in  $\mathfrak{M}$ ,*
- (iv) *if  $F_i(\theta)$  is an monomorphism for each  $i \in I$ , then  $\theta$  is a monomorphism,*
- (v) *if  $F_i(\theta)$  is an epimorphism for each  $i \in I$  and  $M$  is from  $\mathfrak{N}$ , then  $\theta$  is an epimorphism.*

*Then the following consequences hold.*

- (vi) *Any object  $M$  of  $\mathfrak{N}$  is isomorphic to  $\bigoplus_{i \in I} S_i(F_i(M))$ .*
- (vii) *The indecomposable objects in  $\mathfrak{N}$  are of the form  $S_i(A)$  with  $A$  an indecomposable in  $\mathfrak{A}_i$ .*
- (viii) *Given any indecomposable object  $A$  of  $\mathfrak{A}_i$  the object  $S_i(A)$  of  $\mathfrak{M}$  is indecomposable.*
- (ix) *For every  $i, j \in I$ , and every pair of non-zero objects  $A$  and  $A'$  in  $\mathfrak{A}_i$  and  $A'$  in  $\mathfrak{A}_j$ ,  $S_i(A) \cong S_j(A')$  in  $\mathfrak{M}$  iff  $i = j$  and  $A \cong A'$  in  $\mathfrak{A}_i$ .*

*Proof.* This is essentially the due to Gabriel [19]. For a summary see [33, p.22, Lemma].  $\square$

**Application of the lemma.** In the above we will take;  $I = \mathcal{I} = \mathcal{I}(s) \sqcup \mathcal{I}(b)$ ,  $\mathfrak{M} = \mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{Proj})$ ,  $\mathfrak{N} = \mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{proj})$ ,  $\mathfrak{A}_i = k - \mathbf{mod}$  and  $\mathfrak{B}_i = k - \mathbf{Mod}$  for  $i \in \mathcal{I}(s)$ , and  $\mathfrak{A}_i = k[T, T^{-1}] - \mathbf{f.d.mod}$  and  $\mathfrak{B}_i = k[T, T^{-1}] - \mathbf{Mod}$  for  $i \in \mathcal{I}(b)$ . Let us now verify the hypotheses of the above lemma in our setting. (i) follows by lemmas 5.2 and 5.6. (ii) is a corollary of lemma 3.3. (iii) holds by lemmas 5.4 and 5.7. (iv) is the same as lemma 6.1, and (v) is the same as lemma 7.3.

**Proofs of the main theorems.** Theorem 1.1 follows by (vi), (vii) and (viii) in the application of the above lemma to our setting. Theorem 1.2 follows from (ix). Theorem 1.3 follows by theorems 1.1, 6.1, 1.2, and the Krull-Schmidt properties for  $k[T, T^{-1}] - \mathbf{f.d.mod}$  and  $k - \mathbf{mod}$ .

**Derived categories.** In what follows we identify certain subcategories of the homotopy category in order to recover certain classifications of the derived categories, as discussed in the introduction. A *full cycle*  $\alpha_1 \dots \alpha_n$  will refer to a cycle in  $Q$  which is not the product of shorter cycles in  $Q$ , and for which  $\alpha_n \alpha_1 \in \rho$  and  $\alpha_i \alpha_{i+1} \in \rho$  for each  $i \in \{1, \dots, n\}$  such that  $i + 1 \leq n$ . This terminology helps in stating the following.

**Proposition 8.1.** *Let  $N^\bullet = \left( \bigoplus_{\sigma \in \mathcal{S}} P^\bullet(A^\sigma)[-t_\sigma] \right) \oplus \left( \bigoplus_{\beta \in \mathcal{B}} P^\bullet(E^\beta, V^\beta)[-s_\beta] \right)$  be a direct sum of complexes as in the beginning of section 6. Then:*

- (I)  $N^\bullet$  has finitely generated homogeneous components if and only if:
- (Ii) every word  $A^\sigma$  has controlled homogeny,
- (Iii) every  $k[T, T^{-1}]$ -module  $V^\beta$  is finite dimensional, and
- (Iiii) given any  $n \in \mathbb{Z}$ , the number  $\sum_{(B', D') \in \mathcal{W}(s)} \dim(F_{B', D', n}(N^\bullet))$  and the direct sum  $\bigoplus_{(B', D') \in \mathcal{W}(b)} F_{B', D', n}(N^\bullet)$  are both finite.
- (II)  $N^\bullet$  is bounded above (resp. below) if and only if for any word  $A^\sigma$  which occurs the set  $\text{im}(\mu_{A^\sigma})$  is bounded above (resp. below).
- (III) the homology of  $N^\bullet$  is bounded if and only if, for any word  $A^\sigma$  which occurs,  $A^\sigma$  is equivalent to  $B^{-1}D$  where
  - (a) if  $I_B = \mathbb{N}$  there is a full cycle  $\alpha_1 \dots \alpha_n$  such that  $B = B'((\alpha_n^{-1}d_{\alpha_n} \dots \alpha_1^{-1}d_{\alpha_1})^{\pm 1})^\infty$ , and
  - (b) if  $I_D = \mathbb{N}$  there is a full cycle  $\beta_1 \dots \beta_m$  such that  $D = D'((\beta_m^{-1}d_{\beta_m} \dots \beta_1^{-1}d_{\beta_1})^{\pm 1})^\infty$ .

The proof is straightforward for parts (I) and (II) of the above proposition. For part (III), the proof is similar to the proof of lemma 8.2 below. Hence the proof of this proposition was omitted. The classification of objects in  $\mathcal{D}^b(\overline{\Lambda} - \mathbf{mod})$  and  $\mathcal{D}^-(\overline{\Lambda} - \mathbf{mod})$  should now be clear, after applying proposition 8.1, corollary 9.1 and theorems 1.1, 1.2 and 1.3.

**Singularity category.** We start by giving a classification of all acyclic complexes.

**Lemma 8.2.** *If  $P^\bullet$  is an indecomposable acyclic complex of finitely generated projectives, then  $P^\bullet \cong P^\bullet(C)$  where  $C = {}^\infty(\alpha_n^{-1}d_{f(\alpha_n)} \dots \alpha_1^{-1}d_{f(\alpha_1)})^\infty$  for some full cycle  $\alpha_1 \dots \alpha_n$ .*

*Proof.* We know  $P^\bullet$  must be isomorphic to a string complex  $P^\bullet(C)$  or band complex  $P^\bullet(C, V)$  by theorem 1.1. By case analysis one can show  $C$  cannot contain a sub-word of the form  $\gamma^{-1}d_{f(\gamma)}d_{f(\lambda)}^{-1}\lambda$  nor can it contain a sub-word of the form  $d_{f(\gamma)}^{-1}\gamma\lambda^{-1}d_{f(\lambda)}$ . This is because  $P^\bullet$  is acyclic. Hence  $P^\bullet$  must be isomorphic to a string complex  $P^\bullet(C)$  and by symmetry we may assume  $C$  has the required form. The conditions on the arrows are immediate by our conventions on words, the fact that  $P^\bullet$  is acyclic, and lemma 2.1.  $\square$

We now recover a theorem of Kalck in the case  $\overline{\Lambda} = \Lambda$  is a finite dimensional gentle algebra. Let  $D_{sg}(\Lambda)$  denote the singularity category.

**Theorem 8.1.** (Kalck) *Suppose  $Q$  has no linearly oriented cycles so that  $\overline{\Lambda} = \Lambda$  is a finite dimensional gentle algebra. Then there is an equivalence of triangulated categories*

$$D_{sg}(\Lambda) \simeq \coprod_{c \in \mathcal{C}(\Lambda)} D^b(k - \mathbf{mod})/[l(c)]$$

where  $D^b(k - \mathbf{mod})/[l(c)]$  denotes the triangulated orbit category, and  $\mathcal{C}(\Lambda)$  is the set of equivalence classes of repetition-free cyclic paths in  $\Lambda$  with full relations.

*Proof.* By [21, p.5, Theorem 3.4] the finite dimensional gentle algebra  $\Lambda$  is Gorenstein. Hence by [9, p. 16, Theorem 4.4.1] it is enough to describe the full subcategory of  $\mathcal{K}(\Lambda - \mathbf{proj})$  consisting of acyclic complexes. The theorem then follows by lemma 8.2, and as the equivalence relation given on the words  $C = {}^\infty(\alpha_n^{-1}d_{f(\alpha_n)} \dots \alpha_1^{-1}d_{f(\alpha_1)})^\infty$  that classify acyclic complexes corresponds to the equivalence relation on repetition free cyclic paths with full relations from [24].  $\square$

In a paper to follow on from this one the author will discuss some complexes which are K-projective [34] (also known as homotopically projective [25]).

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## 9. APPENDIX: COMPLEXES OF PROJECTIVES WITH RADICAL IMAGES.

For any module finitely generated projective module  $P$  and any  $m \in \mathbb{Z}$  consider the complex  $D^m P^\bullet$  consisting of the module  $P$  in degrees  $m$  and  $m+1$  and zero elsewhere where  $d_{D^m P}^m$  is the identity on  $P$ . Since  $D^m P^\bullet$  is bounded and homotopy equivalent to zero it is zero in the homotopy category (for example see [38, p. 336, Lemma 3.5.44]). So for any collection of finitely generated projective modules  $\{X_n\}_{n \in \mathbb{Z}}$  the complex  $\bigoplus_{n \in \mathbb{Z}} (D^n X_n)^\bullet$  is again an object in  $\mathcal{K}(\overline{\Lambda} - \mathbf{proj})$  isomorphic to zero.

To save space (for cosmetic purposes) we sometimes write morphisms in the form of a matrix (and sometimes the transpose of a matrix). If  $f : A \rightarrow B$  is an epimorphism of finitely generated projective  $\overline{\Lambda}$ -modules then there is an isomorphism  $\alpha : A \rightarrow P_B \oplus Q$  where  $\pi_B : P_B \rightarrow B$  is a projective cover,  $Q$  is projective, and  $(\pi_B \ 0)\alpha = f$ . This follows from the theory of projective covers, which exist as  $\overline{\Lambda}$  is complete and basic (see the remark after definition 1.1). We will use this result the following and refer to it as the remark.

**Lemma 9.1.** *For any complex  $L^\bullet$  of finitely generated projectives and any  $n \in \mathbb{Z}$  there is a finitely generated projective module  $X_n^L$ , a complex  $L_n^\bullet$  of projectives and an isomorphism of complexes  $f_{L,n}^\bullet$  from  $L^\bullet$  to  $L_n^\bullet \oplus (D^n X_n^L)^\bullet$  with the following properties.*

- (i)  $L_n^r = L^r$  and  $f_{L,n}^r = 1_{L^r}$  for  $r \in \mathbb{Z} \setminus \{n, n+1\}$ . (ii)  $d_{L_n}^r = d_L^r$  for  $r \in \mathbb{Z} \setminus \{n-1, n, n+1\}$ .
- (iii)  $\text{im}(d_{L_n}^n) \subseteq \text{rad}(L_n^{n+1})$ .
- (iv) If  $\text{im}(d_L^{n-1}) \subseteq \text{rad}(L^n)$  then  $\text{im}(d_{L_n}^{n-1}) \subseteq \text{rad}(L_n^n)$ . (v) If  $\text{im}(d_L^{n+1}) \subseteq \text{rad}(L^{n+2})$  then  $\text{im}(d_{L_n}^{n+1}) \subseteq \text{rad}(L_n^{n+2})$ .

*Proof.* Let  $W = \text{coker}(d_L^n)$  with the natural map  $\theta : L^{n+1} \rightarrow W$ . By the remark there is a cover  $P_W$  together with the projection  $\pi_1 : P_W \rightarrow W$ , a projective module  $R$  and an isomorphism  $\psi$  which  $(\pi_1 \ 0)\psi = \theta$ . Let  $Z = \ker(\pi_1)$ . Since  $(\pi_1 \ 0)\psi d_L^n = \theta d_L^n = 0$  the map  $\psi d_L^n : L^n \rightarrow Z \oplus R$  is a well defined epimorphism and the restriction of  $\theta \psi^{-1}$  to  $Z \oplus R$  is the restriction of  $(\pi_1 \ 0)$  to  $\ker((\pi_1 \ 0))$ . So any element of  $Y = Z \oplus R$  lies in the image of  $\psi$  restricted to  $\ker(\theta) = \text{im}(d_L^n)$  so  $Y \subseteq \text{im}(\psi d_L^n)$ . By the remark there is a cover  $\pi_3 : P_Y \rightarrow Y$  a module  $R'$  and an isomorphism  $\omega : L^n \rightarrow P_Y \oplus R'$  for which  $(\pi_3 \ 0)\omega = \psi d_L^n$ . Consider the cover  $\pi_2 : P_Z \rightarrow Z$  of  $Z$ . By [26, p.351, Examples 24.11 (3)]  $P_Y = P_Z \oplus R$  and  $\pi_3 = \pi_2 \oplus 1_R$ . So we can write  $\omega = (\omega_1 \ \omega_2 \ \omega_3)^t$  and therefore and so  $\psi d_L^n = (\pi_2 \omega_1 \ \omega_2)$ . We now have that the diagram

$$\begin{array}{ccccc} L^n & \xrightarrow{\quad d_L^n \quad} & L^{n+1} & & \\ (\omega_1 \ \omega_2 \ \omega_3)^t \downarrow & & \downarrow \psi & & \\ P_Z \oplus R' \oplus R & \xrightarrow{(\pi_2 \ 0) \oplus 1_R} & Z \oplus R & \xrightarrow{\subseteq} & P_W \oplus R \end{array}$$

commutes. Let  $\alpha = \iota \pi_2$  where  $\iota$  is the inclusion of  $Z = \ker(\pi_1)$  in  $P_W$ . Note that as  $\omega$  is an isomorphism it has an inverse  $\eta = (\eta_1 \ \eta_2 \ \eta_3)$ . Since  $\pi_1 : P_W \rightarrow W$  is a projective cover and so  $Z = \ker(\pi_1)$  is a superfluous sub-module of  $P_W$ , we have that (say by [26, p. 348, Proposition 24.4 ])  $\text{im}(\alpha) \subseteq Z$  which lies in  $\text{rad}(P_W)$ . Let  $X_n^L = R$ ,  $L_n^{n+1} = P_W$ ,  $L_n^n = P_Z \oplus R'$ , and  $d_{L_n}^n = (\alpha \ 0)$ . The above diagram can now be rewritten as

$$\begin{array}{ccccccc} L^{n-1} & \xrightarrow{d_M^{n-1}} & L^n & \xrightarrow{d_M^n} & L^{n+1} & \xrightarrow{d_M^{n+1}} & L^{n+2} \\ \parallel & & \downarrow \varphi & & \downarrow \psi & & \parallel \\ L^{n-1} & \xrightarrow[\varphi d_L^{n-1}]{(\gamma \ \delta)^t} & L_n^n \oplus X_n^L & \xrightarrow[\psi d_L^n \varphi^{-1}]{d_{L_n}^n \oplus 1_{X_n^L}} & L_n^{n+1} \oplus X_n^L & \xrightarrow[\lambda \mu]{d_L^{n+1} \psi^{-1}} & L^{n+2} \end{array}$$

The commutativity of this diagram gives  $d_{L_n}^n \gamma = 0$  and  $\delta = 0$  since  $\psi d_L^n d_L^{n-1} = 0$ . Similarly  $\lambda d_{L_n}^n = 0$  and  $\mu = 0$ . Set  $L_n^r = L^r$  and  $f_{L,n}^r = 1_{L^r}$  for  $r \in \mathbb{Z}$  with  $r \neq n, n+1$ ,  $f_{L,n}^n = \varphi$ ,  $f_{L,n}^{n+1} = \psi$ ,  $d_{L_n}^s = d_L^s$  for  $s \in \mathbb{Z}$  with  $s \neq n-1, n, n+1$ ,  $d_{L_n}^r = \lambda$ , and  $d_{L_n}^{n-1} = \gamma$ . By construction  $L_n^\bullet$  defines a complex and  $f_{L,n}^\bullet$  defines an isomorphism between  $L^\bullet$  and the direct sum of  $L_n^\bullet$  and the complex  $(D^n X_n^L)^\bullet$ . The required properties hold by construction.  $\square$

We switch to the more convenient notation for the remainder of this section. Fix an integer  $m \leq -1$  and a complex  $M^\bullet$  of finitely generated projective  $\overline{\Lambda}$ -modules. Applying lemma 9.1  $-m$ -times yields complexes of finitely generated projectives  $M_{[0]}^\bullet = M^\bullet$ ,  $M_{[-1]}^\bullet = M_{-1}^\bullet$  up to  $M_{[m]}^\bullet = (M_{[m+1]}^\bullet)_m^\bullet$  (defined iteratively).

Let  $A_0 = 0$  and for  $m \leq t \leq -1$  let  $A_t = X_t^{M_{[t+1]}^\bullet}$  and  $N_t^\bullet = M_{[t]}^\bullet \oplus (D^t A_t)^\bullet \oplus \cdots \oplus (D^{-1} A_{-1})^\bullet$ . Along with the complexes  $M_{[t]}^\bullet$  there are isomorphisms  $f_{[t]}^\bullet = f_{M_{[t+1]}, t}^\bullet$  from  $M_{[t+1]}^\bullet$  to  $M_{[t]}^\bullet \oplus (D^t A_t)^\bullet$ . Let  $p_m^\bullet : M^\bullet \rightarrow N_m^\bullet$  be the composition  $g_m^\bullet \cdots g_{-1}^\bullet$  where  $g_{-1}^\bullet = f_{[-1]}^\bullet$  and  $g_t^\bullet = f_{[t]}^\bullet \oplus 1_{D^{t+1} A_{t+1}} \oplus \cdots \oplus 1_{D^{-1} A_{-1}}$  for any  $t$  with  $m \leq t < -1$ .

Now let  $r \in \mathbb{Z}$  be arbitrary. For  $r \leq 0$  let  $M_-^r = M_{[r-1]}^r$ ,  $d_{M_-}^r = d_{M_{[r-1]}}^r$  and  $f_-^r = p_{r-1}^r$ . Let  $N_-^0 = M_-^0$  and  $d_{N_-}^0 = (d_{M_-}^0 \ 0 \ 0)$  and for  $r \leq -1$  let  $N_-^r = M_-^r \oplus A_{r-1} \oplus A_r$  and

$$d_{N_-}^r = d_{M_-}^r \oplus \begin{pmatrix} 0 & 1_{A^r} \\ 0 & 0 \end{pmatrix} : N_-^r = M_-^r \oplus A_{r-1} \oplus A_r \longrightarrow M_-^{r+1} \oplus A_r \oplus A_{r+1} = N_-^{r+1}.$$

For  $r \geq 1$  let  $M_-^r = N_-^r = M^r$ ,  $d_{M_-}^r = d_{N_-}^r = d_M^r$  and  $f_-^r = 1_{M^r}$ . Note that  $d_{N_-}^{-1}$  has co-domain  $M_-^0 \oplus A_{-1} = M_{[-1]}^0 \oplus X_{-1}^{M_{[0]}^\bullet} = M^0 \oplus X_{-1}^M$ . By definition (and some simple matrix multiplication)  $d_{N_-}^{r+1} d_{N_-}^r = 0$ . There are several consequences of the lemma above for the updated notation.

**Lemma 9.2.** *In the above notation;*

- (i)  $M_-^\bullet \oplus \bigoplus_{n \leq -1} (D^{-n} A_{-n})^\bullet = N_-^\bullet$ .
- (ii)  $f_-^\bullet$  defines an isomorphism of complexes from  $M^\bullet$  to  $N_-^\bullet$ .
- (iii)  $\text{im}(d_{M_-}^r) \subseteq \text{rad}(M_-^{r+1})$  for each  $r \in \mathbb{Z}$  for which  $r \leq -1$ .

*Proof.* By construction and lemma 9.1. □

A dual result follows which will require some similar notation. This time fix  $n \in \mathbb{Z}$  with  $n \geq 0$  and apply lemma 9.1  $n$ -times starting at degree 0 to find a collection of complexes of finitely generated projectives  $M_{(0)}^\bullet = M^\bullet$ ,  $M_{(1)}^\bullet = (M_{(0)}^\bullet)_1$  up to  $M_{(n-1)}^\bullet = (M_{(n-1)}^\bullet)_n^\bullet$ .

For each  $t \in \mathbb{Z}$  with  $n \geq t \geq 0$  let  $B_t = X_t^{M_{(t-1)}^\bullet}$  and  $N_t^\bullet = M_{(t)}^\bullet \oplus (D^{(t)} B_t)^\bullet \oplus \cdots \oplus (D^0 B_0)^\bullet$ . There are isomorphisms  $f_{(t)}^\bullet = f_{M_{(t-1)}, t}^\bullet$  from  $M_{(t-1)}^\bullet$  to  $M_{(t)}^\bullet \oplus (D^t A_t)^\bullet$ . Let  $q_n^\bullet : M^\bullet \rightarrow N_n^\bullet$  be the composition  $h_n^\bullet \cdots h_0^\bullet$  where  $h_0^\bullet = f_{(0)}^\bullet$  and  $h_t^\bullet = (f_{(n)}^\bullet \oplus 1_{D^{n-1} B_{n-1}} \oplus \cdots \oplus 1_{D^0 B_0}) \circ \cdots \circ f_{(0)}^\bullet$  whenever  $t > 0$ .

Now let  $r \in \mathbb{Z}$  be arbitrary. For  $r \geq 0$  let  $M_+^r = M_{(r+1)}^r$ ,  $d_{M_+}^r = d_{M_{(r+1)}}^r$  and  $f_+^r = q_r^r$ . Let  $N_+^0 = M_+^0$  and  $d_{N_+}^0 = (d_{M_+}^0 \ 0 \ 0)$  and for  $r \geq 1$  let  $N_+^r = M_+^r \oplus B_r \oplus B_{r-1}$  and

$$d_{N_+}^r = d_{M_+}^r \oplus \begin{pmatrix} 0 & 0 \\ 1_{B_r} & 0 \end{pmatrix} : N_+^r = M_+^r \oplus B_r \oplus B_{r-1} \longrightarrow M_+^{r+1} \oplus B_{r+1} \oplus B_r = N_+^{r+1}$$

For  $r \leq -1$  let  $M_+^r = N_+^r = M^r$ ,  $d_{M_+}^r = d_{N_+}^r = d_M^r$  and  $f_+^r = 1_{M^r}$ . There are similar consequences as above. Here we observe an additional consequence that will serve a purpose later.

**Lemma 9.3.** *In the above notation;*

- (i)  $M_+^\bullet$  and  $N_+^\bullet$  define complexes of finitely generated projectives and  $M_+^\bullet \oplus \bigoplus_{n \in \mathbb{N}} (D^n B_n)^\bullet = N_+^\bullet$ .
- (ii)  $f_+^\bullet$  defines an isomorphism of complexes from  $M^\bullet$  to  $N_+^\bullet$ . (iii)  $\text{im}(d_{M_+}^r) \subseteq \text{rad}(M_+^{r+1})$  for each  $r \in \mathbb{Z}$  for which  $r \geq 0$ .
- (iv) For each  $r \in \mathbb{Z}$  with  $r \leq -1$  we have  $\text{im}(d_{M_+}^r) \subseteq \text{rad}(M_+^{r+1})$  given  $\text{im}(d_M^r) \subseteq \text{rad}(M^{r+1})$ .

*Proof.* Similar to lemma 9.2. □

We now see a proof of proposition 1.1 which was adapted from [4, p.12, Proposition 3.1].

**Corollary 9.1.** *The inclusion of  $\mathcal{K}_{\text{rad}}(\overline{\Lambda} - \mathbf{proj})$  into  $\mathcal{K}(\overline{\Lambda} - \mathbf{proj})$  is dense.*

*Furthermore there is a functor from  $\mathcal{K}_{\text{rad}}^-(\overline{\Lambda} - \mathbf{proj})$  to  $\mathcal{D}^-(\overline{\Lambda} - \mathbf{mod})$  which is full, faithful, dense, and additive; and restricts to a functor from  $\mathcal{K}_{\text{rad}}^{-,b}(\overline{\Lambda} - \mathbf{proj})$  to  $\mathcal{D}^b(\overline{\Lambda} - \mathbf{mod})$ .*

*Proof.* Let  $(M_{-}^{\bullet})_{+}^{\bullet} = N^{\bullet}$ . By lemmas 9.2 and 9.3 there is an isomorphism between  $M^{\bullet}$  and  $N^{\bullet}$  in  $\mathcal{K}(\overline{\Lambda} - \mathbf{proj})$  and  $N^{\bullet}$  is a complex of finitely generated projectives with radical images. The second part of the corollary follows from the above and [38, p. 333, Proposition 3.5.43].  $\square$

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